MULTIPLICATIVE PROGRAMMING: THEORY AND ALGORITHMS

By

GEORGE BOGER

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By

George Boger

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Multiplicative programming problems are mathematical optimization problems in which the objective function contains a product of several real valued functions defined over a common domain and the feasible decisions are described by a nonempty set. These optimization problems have some important applications in engineering, finance, economics, and other fields. Multiplicative programming problems, however, are difficult global optimization problems that are known to be NP-hard.

This dissertation has two purposes. The first is to develop and test a heuristic algorithm that finds a good solution, though not necessarily a globally optimal solution, for the linear multiplicative programming problem. The second purpose is to develop a global solution algorithm for the linear multiplicative programming problem that is potentially more efficient than existing algorithms for this problem.

To evaluate the effectiveness in practice of the heuristic algorithm, we have written a FORTRAN computer program and used it to solve 260 randomly generated linear multiplicative programming problems of various sizes. Our experimental results show that the computational requirements of the heuristic algorithm are not overly burdensome when compared to the effort required to solve a linear multiplicative programming problem.

The framework of the outcome-space, cutting-plane algorithm is taken from a pure cutting plane, decision set-based method developed by Horst and Tuy for solving concave minimization problems. By adapting the approach of this method to an outcome-space reformulation of the linear multiplicative programming problem, rather than directly applying the method to the original decision set formulation, it is expected that considerable computational savings can potentially be obtained. We also show how additional computational benefits might be obtained by implementing the new algorithm appropriately. To illustrate the new algorithm, we apply it to the solution of a sample problem.

CHAPTER 1
INTRODUCTION

1.1. The Multiplicative Programming Problem

Multiplicative programming problems are mathematical optimization problems in which the objective function contains a product of several real valued functions defined over a common domain and the feasible decisions are describe by a nonempty set. These problems occur is a wide variety of application areas.

For example, Konno and Inori (1989) studied a bond portfolio optimization problem in which the portfolio's performance is measured by a number of indices such as the average coupon rate, the average terminal yield, and the average length to maturity. The goal of the portfolio manager is to improve the performance of the portfolio by purchasing or selling bonds in the marketplace subject to some limiting constraints. The manager must consider multiple incomparable objectives such as maximizing the average terminal yield and minimizing the average maturity time. Konno and Inori choose to optimize several objectives simultaneously by multiplying them together since the objectives do not share a common scale.

Another example of a multiplicative programming problem, given in Maling, Mueller and Heller (1982), is a packaging problem encountered in designing very large-scale integrated circuit (VLSI) chips and laying out building floor plans or manufacturing plant facilities. In the problem, the overall rectangular dimensions of the feasible layout

plans are constrained rather than fixed. Different layout plans with differing overall rectangular dimensions are obtained according to how the components of a system are arranged within each plan. The objective is to find the arrangement of components that minimizes the overall layout area subject to certain constraints on the area and the perimeter of the layout.

Henderson and Quandt (1971, p. 15) also give an application of multiplicative programming problems. Their example is from microeconomics. In their example, a rational consumer wishes to find a combination of two commodities to purchase from which he will derive the highest possible level of satisfaction. Budgetary constraints and the availability of the commodities limit the quantities the consumer may purchase. The consumer's level of satisfaction is captured by his utility function, which is assumed to be the product of the quantities of the two commodities. The rational consumer's problem is then formulated as maximizing his utility function subject to the budgetary and commodity availability constraints.

The multiplicative programming problem or, more briefly, the multiplicative program, may be formulated mathematically as

$$(P_X) \qquad \min h(x) = \prod_{j=1}^{p} f_j(x), \text{ s.t. } x \in X,$$

where $p \geqq 2$ is an integer, $X \subseteq \boldsymbol{R}^n$, and, for each $j = 1, 2, \ldots, p$, $f_j : X \to \boldsymbol{R}$ satisfies $f_j(x) \geqq 0$ for all $x \in X$. For simplicity we will assume throughout this proposal that the minimum of problem (P_X) is achieved at some point $x^* \in X$. In addition we will assume that p is significantly less than n since this holds for virtually all applications of

multiplicative programming problems. If $f_j(\bar{x}) = 0$ for some $j \in \{1, 2, \ldots, p\}$ and some $\bar{x} \in X$, then clearly $\bar{x}$ is a global optimal solution. This condition can be checked by solving p minimization problems $\min \{f_j(x) \mid x \in X\}$, $j = 1, 2, \ldots, p$. Therefore, we may assume without loss of generality that, for each $j = 1, 2, \ldots, p$, $f_j(x) > 0$ holds for all $x \in X$.

The objective function h of problem (P_X) is generally not a convex function. As a result, problem (P_X) belongs to a class of nonconvex programming problems called global optimization problems. In contrast to convex programming problems, there may be many local minima for problem (P_X) that are not globally optimal. Conventional local optimization methods based on gradients, subgradients, conjugate directions, or the Karush-Kuhn-Tucker conditions, for instance, are at best guaranteed only to find a local minimum. These methods must then terminate, since there is neither a local criterion for certifying the global optimality of a given solution nor a way to determine how to proceed to a better solution if the solution is not globally optimal. From the perspective of computational complexity, problem (P_X) is a difficult problem that is known to be NP-hard even when the objective function is simply $h(x) = x_1 \cdot x_2$ and the feasible region X is a polyhedron (Matsui 1996).

When in addition to the assumptions given previously for problem (P_X), X is a convex set and, for each $j = 1, 2, \ldots, p$, $f_j : X \to \boldsymbol{R}$ is a concave function, we obtain the concave case of problem (P_X), called the concave multiplicative programming problem. The convex case of problem (P_X), called the convex multiplicative programming

problem, is obtained when, in addition to the assumptions made previously for problem (P_X), X is a convex set and, for each $j = 1, 2, \ldots, p$, $f_j : X \to R$ is a convex function. A special linear case of problem (P_X), called the linear multiplicative programming problem, is obtained when, in addition to the assumptions make previously for problem (P_X), X is a compact polyhedron and, for each $j = 1, 2, \ldots, p$, $f_j : X \to R$ is a linear function (Konno and Kuno 1992).

1.2. Reformulations of the Multiplicative Programming Problem

During the 1990's there has been a resurgence of interest in problem (P_X). Encouraged by the rapid advances in high speed computing, researchers began developing and testing new methods for solving global optimization problems that arise in practical applications, including problem (P_X).

Included among the global optimization methods used to solve problem (P_X) for the special case when $p = 2$ are various parametric simplex method-based algorithms (e.g., Konno and Kuno 1992, Konno and Kuno 1995, Konno, Yajima, and Matsui 1991, and Schaible and Sodini 1995), branch and bound procedures (e.g., Kuno 1996 and Muu and Tam 1992), and various other types of algorithms (e.g., Konno and Kuno 1990, Pardalos 1990, and Tuy and Tam 1992).

When $p > 2$, globally solving problem (P_X) has been shown empirically to require considerably more computational effort than when $p = 2$ (see, e.g., Ryoo and Sahinidis 1996). A smaller number of the algorithms for solving problem (P_X) when

$p > 2$ solve the problem directly without reformulating it as an outcome-space problem. Included among these, for instance, is the polyhedral annexation algorithm of Tuy (1991).

Most of the algorithms for solving problem (P_X) when $p > 2$, however, solve the problem indirectly by globally solving an outcome-space reformulation of the problem instead. This is because in practical applications p is routinely much smaller than n, often by two or more orders of magnitude. As a result, working in $\boldsymbol{R}^p$ is computationally less challenging than working in $\boldsymbol{R}^n$.

Let $y \in \boldsymbol{R}^p$ denote the p-vector with jth entry equal to y_j, $j = 1, 2, \ldots, p$. For each $j = 1, 2, \ldots, p$, let $\hat{y}_j \in \boldsymbol{R}$ satisfy

$$\hat{y}_j \geqq \sup f_j(x), \text{ s.t. } x \in X,$$

where $\hat{y}_j = +\infty$ is possible, and let $\hat{y} \in \boldsymbol{R}^p$ denote the vector with jth entry equal to $\hat{y}_j$, $j = 1, 2, \ldots, p$. Let $f(x)$ denote the vector $f(x) = [f_1(x), f_2(x), \ldots, f_p(x)]^T$, where $f_j : X \to \boldsymbol{R}$, $j = 1, 2, \ldots, p$, are the functions used in defining problem (P_X). Thoai (1991) and later Konno and Kuno (1995) based their outer approximation algorithms for respectively solving the convex and linear cases of problem (P_X) on one of the more direct reformulations of problem (P_X) as an outcome-space problem. Their reformulation is given by

$$(P_{Y^{\leqq}}) \qquad \min \prod_{j=1}^{p} y_j, \text{ s.t. } y \in Y^{\leqq},$$

where

$$Y^{\leqq} = \left\{ y \in \boldsymbol{R}^p \,\middle|\, f(x) \leqq y \leqq \hat{y} \text{ for some } x \in X \right\}.$$

Falk and Palocsay (1994) based their branch and bound, image space algorithm for the linear case of problem (P_X) on another outcome-space reformulation that is closely related to problem $(P_{Y^{\leq}})$. Their reformulation is given by

$$(P_Y) \qquad \min \prod_{j=1}^{p} y_j, \quad \text{s.t. } y \in Y,$$

where

$$Y = \left\{ y \in R^p \,\middle|\, y = Cx \text{ for some } x \in X \right\}$$

and C is a $(p \times n)$ matrix whose rows are c_j^T, $j = 1, 2, \ldots, p$.

1.3. Purpose and Organization of the Dissertation

This dissertation has two main purposes. The first is to develop and test a heuristic algorithm that finds a good solution, though not necessarily a globally optimal solution, for the linear case of problem (P_X). The second purpose is to develop an exact global solution algorithm for the linear case of problem (P_X) that is potentially more efficient than existing algorithms for this problem.

Since the linear multiplicative programming problem is known to be an NP-hard, multiextremal global optimization problem, it is inherently more difficult to globally solve than a convex programming problem of the same size. In some application cases, a solution will adequately meet the requirements of a user; see, e.g., Konno and Inori (1989). In these cases, the use of a heuristic algorithm seems to be appropriate for finding a satisfactory solution. To date, however, there is no known heuristic algorithm tailored to finding a good solution for the linear multiplicative programming problem. In their review of algorithms for solving problem (P_X), Konno and Kuno (1995) do not mention

any heuristic algorithms for problem (P_X), and our survey of the literature has revealed none.

To develop the heuristic algorithm, we first analyze the concave multiplicative programming problem. The analysis yields a new way to write a concave multiplicative programming problem as a concave minimization problem. As a result, a concave multiplicative programming problem can be solved by using any existing concave minimization algorithm without resorting to a reformulation of the problem. We also show that some relationships exist between concave multiplicative programming problems and certain multiple-objective mathematical programs. These relationships are exploited to develop the heuristic algorithm for the linear case of problem (P_X).

For cases where a linear multiplicative program must be solved for an exact global optimal solution, we expect that globally solving the outcome-space reformulation $(P_{Y^{\leq}})$ instead will result in a significant decrease in the computational effort over that required to directly solve the problem. This is because in typical applications of linear multiplicative programs, p is several orders of magnitude smaller than n. As a result, working in R^p should be computationally less challenging than working in R^n.

To globally solve the outcome-space reformulation $(P_{Y^{\leq}})$ of a linear multiplicative program, we develop an outcome-space, pure cutting plane algorithm that works in R^p. The framework for the algorithm is taken from a pure cutting plane, decision set-based concave minimization method developed by Horst and Tuy (1993). We show how to adapt this method to solving the reformulation $(P_{Y^{\leq}})$ of a linear multiplicative program for a global extreme point optimal solution. Once this global solution is found, we can

recover a globally optimal solution for the linear multiplicative program in decision space. As a further computational enhancement, we also show that for purposes of implementation, the mechanics of the outcome-space, cutting-plane algorithm can be applied to the smaller problem (P_Y) instead of problem $(P_{Y^{\leqq}})$.

The organization of the proposal is as follows. In Chapter 2 we present a review of the literature on multiplicative programming problems. In Chapter 3 we analyze the concave multiplicative programming problem, apply the results to develop a heuristic algorithm for the linear multiplicative programming problem, and report test results using the heuristic algorithm on some randomly-generated problems. In Chapter 4 we analyze the reformulation problem $(P_{Y^{\leqq}})$ and show that, under certain convexity assumptions on $Y^{\leqq}$, problem $(P_{Y^{\leqq}})$ has a global extreme point optimal solution $y^* \in Y^{\leqq}$. We then present a procedure that is guaranteed to find a strict local optimal extreme point solution for the reformulation problem $(P_{Y^{\leqq}})$ of the linear multiplicative program. In Chapter 5 we present an outcome-space, cutting-plane algorithm for globally solving a linear multiplicative program. The algorithm employs the strict local optimal search procedure presented in Chapter 4. We also illustrate the algorithm by applying it to the solution of a sample problem. Finally, in Chapter 6, we give an overall summary and conclusions, and we discuss directions for further research.

CHAPTER 2
A REVIEW OF THE LITERATURE ON MULTIPLICATIVE PROGRAMMING PROBLEMS

2.1. Organization of the Literature Review

In this chapter we present a review of the literature on methods proposed for solving multiplicative programming problems. The only known literature review on multiplicative programming problems appears in Konno and Kuno (1995). In their literature review Konno and Kuno defined multiplicative programming problems as "a class of minimization problems containing a product of several convex functions either in its objective function or in its constraints." They included problems in which the objective function contained the summation of a convex function and the product of convex functions.

Konno and Kuno (1995) organized their literature review based on whether the problem data are linear or nonlinear and on the number of functions that appear in the objective function. They considered solution methods for the following multiplicative programming problems.

The first multiplicative programming problem considered by Konno and Kuno is the special case of quadratic programming

$$\text{(LMP2)} \qquad \min f(x) = \left(\left\langle c^1, x\right\rangle + d_1\right)\left(\left\langle c^2, x\right\rangle + d_2\right), \text{ s.t. } x \in D,$$

where $D := \left\{x \in R^n \middle| Ax \geqq b,\ x \geqq 0\right\}$ is a non-empty polytope (bounded polyhedron) in

which A is an $m \times n$ matrix, $b \in \boldsymbol{R}^m$, and, for each $i = 1, 2$, $c^i \in \boldsymbol{R}^n \setminus \{0\}$ and $d_i \in \boldsymbol{R}$. In addition, it is assumed that, for each $x \in D$, $\langle c^i, x \rangle + d_i > 0$, $i = 1, 2$.

The second multiplicative programming problem that they considered is the convex multiplicative programming problem

$$\text{(CMP)} \qquad \min f(x) = \prod_{j=1}^{p} f_j(x), \text{ s.t.} x \in X,$$

where $X \subseteq \boldsymbol{R}^n$ is a nonempty, compact, convex set and, for each $j = 1, 2, \ldots, p$, $f_j : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_j(x) > 0$ for all $x \in X$.

Konno and Kuno (1995) considered two special cases of problem (CMP): (1) the case where $p = 2$ and (2) the case where $p \geqq 2$ and the problem data are linear. The second case may be defined as the following extension of problem (LMP2):

$$\text{(LMP)} \qquad \min f(x) = \prod_{i=1}^{p} \left[\langle c^i, x \rangle + d_i \right], \text{ s.t.} x \in D,$$

where $p \geqq 2$ is an integer and, for each $i = 1, 2, \ldots, p$, $\langle c^i, x \rangle + d_i > 0$ holds for all $x \in D$.

Finally, Konno and Kuno (1995) considered three classes of problems related to problem (CMP). In the first class is the following problem:

$$\text{(GCMP)} \quad \min f(x) = f_0(x) + \sum_{j=1}^{q} f_{2j-1}(x) f_{2j}(x), \text{ s.t.} x \in X,$$

where, for each $j = 0, 1, \ldots, 2q$, $f_j : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_j(x) > 0$ for all $x \in X$.

The second class is a special case of (GCMP) in which $q = 1$ and the problem data are linear. This class may be defined as the following extension of problem (LMP2):

$$\text{(GLMP)}\quad \min f(x) = \langle c^0, x\rangle + \left(\langle c^1, x\rangle + d_1\right)\left(\langle c^2, x\rangle + d_2\right),\ \text{s.t.}\ x \in D,$$

where $c^0 \in \boldsymbol{R}^n$ and $c^i, d_i,\ i = 1, 2,$ and D are defined as in problem (LMP2).

The third class of problems considered by Konno and Kuno (1995) is the minimization of a convex function over a feasible region that includes a product of convex functions in its constraint set.

Konno and Kuno's coverage of the literature is not exhaustive. They focused on algorithms that have been demonstrated by computational experiments to be practical for reasonably large problems (Konno and Kuno 1995, p. 370). Algorithms proposed by Konno, Kuno, and their associates have been tested on randomly generated problems and the results reported. However, computational results have not been reported by most of the other researchers and therefore their methods were not included in the review.

Since the publication of the review by Konno and Kuno, two more multiplicative programming problems have been discussed in the literature. The first problem adds a convex function to the objective of problem (LMP2) to obtain the problem:

$$\text{(CLMP)}\quad \min f(x) = g(x) + \left(\langle c^1, x\rangle + d_1\right)\left(\langle c^2, x\rangle + d_2\right),\ \text{s.t.}\ x \in D,$$

where $g : \boldsymbol{R}^n \to \boldsymbol{R}$ is a twice differentiable convex function and $c^i, d_i,\ i = 1, 2,$ and D are defined as in problem (LMP2). The second problem adds a convex function to problem (CMP) to obtain the problem:

$$\text{(CCMP)}\quad \min f(x) = f_0(x) + \prod_{j=1}^{p} f_j(x),\ \text{s.t.}\ x \in X,$$

where $f_0 : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_0(x) > 0$ for all $x \in X$ and f_j, $j = 1, 2, \ldots, p,$ and X are defined as in problem (CMP).

The emphasis of this review will be on optimization problems in which a product of functions appears in the objective function. Optimization problems with objective functions that are comprised of a summation of a function and the product of functions are also included in the review. Methods proposed for solving these problems may be adapted to solve a problem whose objective function is strictly a product of functions by setting the added function to the null function. The functions that appear in the objective function will be either convex or linear functions since to date these are the only multiplicative programming problems to appear in the literature. In this review we will not consider optimization problems in which a product of functions appears in the constraint set.

Like the review of Konno and Kuno (1995), this literature review is organized based on whether the problem data are linear or nonlinear and on the number of functions that appear in the objective function. It is divided into the following four sections. Section 2.2 reviews the methods proposed to solve problems (LMP2), (GLMP), and (CLMP). Section 2.3 reviews the methods to solve problem (LMP) that are extensions of methods for problem (LMP2). Section 2.4 reviews the methods to solve problems (CMP), (GCMP), and (CCMP). Section 2.5 reviews the methods to solve problem (LMP) as a concave minimization problem.

The rationale for organizing the literature review in this way is as follows. Historically, the first algorithms for solving multiplicative programming problems were specifically proposed for solving problem (LMP2). Problems (GLMP) and (CLMP) are grouped with problem (LMP2) since they were conceived as extensions of that problem. Several of the algorithms proposed for solving problem (LMP2) can be extended to solve

the problem (LMP), since they do not depend upon having only two functions in the product term of the objective function. Problems (LMP2), (LMP), (GLMP), and (CLMP) contain linear functions and polyhedral feasible regions. Algorithms for solving these problems are implemented with the aid of the simplex method, which is used to solve linear programming subproblems. The problems (CMP), (GCMP), and (CCMP) contain nonlinear data and must rely on other optimization methods to solve nonlinear convex programming problems. The latter three problems are therefore placed in a separate group. Problems (GCMP) and (CCMP) are included in the group with problem (CMP) because only one article addresses each problem, and they were conceived as extensions of problem (CMP). Finally, two articles appeared in the literature that proposed solving problem (LMP) as a concave minimization problem using techniques that the authors had previously developed.

Table 2.1 gives a summary of the multiplicative programming problems considered in this literature review along with the assumptions placed on the feasible region and the objective function of each problem.

2.2. Methods to Solve Problems (LMP2), (GLMP), and (CLMP)

The methods for solving problem (LMP2), (GLMP), and (CLMP) are further divided into four categories. In the first category are those methods that analyze problem (LMP2) as a special case of quadratic programming. In the second category are algorithms that analyze problem (LMP2) by searching the outcome set. In the third category are the algorithms that solve an easier parametric programming problem rather than directly solving problems (LMP2), (GLMP), and (CLMP). In the last category are

Table 2.1. Summary of Multiplicative Program Types and Assumptions on Problems

Problem	Assumptions on the Feasible Region	Objective Function	Assumptions on the Objective Function
LMP2	D is a bounded polyhedron.	$\left(\langle c^1, x\rangle + d_1\right)\left(\langle c^2, x\rangle + d_2\right)$	$\langle c^i, x\rangle + d_i > 0,\ i = 1, 2$, for all $x \in D$.
GLMP	D is a bounded polyhedron.	$\langle c^0, x\rangle + \left(\langle c^1, x\rangle + d_1\right)\left(\langle c^2, x\rangle + d_2\right)$	$\langle c^0, x\rangle > 0$ and $\langle c^i, x\rangle + d_i > 0,\ i = 1, 2$, for all $x \in D$.
CLMP	D is a bounded polyhedron.	$g(x) + \left(\langle c^1, x\rangle + d_1\right)\left(\langle c^2, x\rangle + d_2\right)$	$g : \boldsymbol{R}^n \to \boldsymbol{R}$ is a twice differentiable convex function and $\langle c^i, x\rangle + d_i > 0,\ i = 1, 2,$ for all $x \in D$.
LMP	D is a bounded polyhedron.	$\prod_{i=1}^{p}\left[\langle c^i, x\rangle + d_i\right]$	$\langle c^i, x\rangle + d_i > 0,\ i = 1, 2, \ldots, p,$ for all $x \in D$.
CMP	X is a compact convex set.	$\prod_{j=1}^{p} f_j(x)$	For each $j = 1, 2, \ldots, p,\ f_j : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_j(x) > 0$ for all $x \in X$.
GCMP	X is a compact convex set.	$f_0(x) + \sum_{j=1}^{p} f_{2j-1}(x) f_{2j}(x)$	For each $j = 0, 1, \ldots, p,\ f_j : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_j(x) > 0$ for all $x \in X$.
CCMP	X is a compact convex set.	$f_0(x) + \prod_{j=1}^{p} f_j(x)$	For each $j = 0, 1, \ldots, p,\ f_j : \boldsymbol{R}^n \to \boldsymbol{R}$ is a convex function that satisfies $f_j(x) > 0$ for all $x \in X$.

two algorithms that solve problem (LMP2) based on the method of polyhedral annexation.

2.2.1. Methods Based on Quadratic Programming

Since the objective function of problem (LMP2) can be expressed as

$$f(x) = \left(\left\langle c^1, x\right\rangle + d_1\right)\left(\left\langle c^2, x\right\rangle + d_2\right) = \frac{1}{2}x^T Q x + r^T x + d_1 d_2,$$

where $r \in R^n$, and Q is a real symmetric $n \times n$ matrix, problem (LMP2) is a special class of quadratic programming. Swarup (1966a and 1966b) was the first researcher to analyze problem (LMP2) in this way, but he did not propose any exact solution algorithms. His two articles are included in the literature review for completeness. Pardalos (1990) also analyzed problem (LMP2) in this way, and he proposed an exact global solution algorithm.

Swarup (1966a) showed that if both linear functions $\left\langle c^i, x\right\rangle + d_i$, $i = 1, 2$ are positive over the feasible region D, the objective function f is quasiconcave over D, It is well known that generally for any local minimizer of a quasiconcave function over a polytope, there exists an extreme point local minimizer over the polytope that has the same function value. Swarup proposed a simplex based method for finding such a local optimal solution. The key to the algorithm is a test that determines if entering a given nonbasic variable into the current simplex basis will lower the objective function value. A simplex basis of a local optimal solution can be reached by beginning at any feasible basis and moving through a sequence of simplex tableaux by pivoting in qualifying nonbasic variables until none remain. Once a local optimal solution is found, the

algorithm stops. No information is available to either certify the global optimality of the solution or to determine how to proceed to an improved solution.

In another work, Swarup (1966b) formulated the following parametric linear program by introducing an auxiliary variable ξ and moving one of the linear functions into the constraint set:

$$\text{(MP1)} \qquad \min F(x;\xi) = \langle c^1, x \rangle + d_1,$$

$$\text{s.t.} \quad x \in D,$$

$$\langle c^2, x \rangle + d_2 = \xi, \; \xi \geqq 0.$$

Since $\langle c^2, x \rangle + d_2$ appears in the constraint set, dual pricing information is available to determine the value of $\langle c^1, x \rangle + d_1$ as ξ is set to achievable values of $\langle c^2, x \rangle + d_2$ over D. Swarup derived a test that uses this information to determine when ξ is set to a level that corresponds to a local optimal solution. All local optimal solutions can then theoretically be found by parametrically solving problem (MP1) over all achievable values of ξ. A global optimal solution x^* of problem (LMP2) can then be found by identifying a global solution (x^*, ξ^*) of problem (MP1).

Pardalos (1990) observed that if c^1 and c^2 are linearly independent, then the Hessian matrix Q of the objective function of problem (LMP2) has one positive eigenvalue and one negative eigenvalue, and the remaining eigenvalues are equal to zero. By applying the spectral decomposition theorem of linear algebra, the objective function can be rewritten in terms of two variables. The problem can then be solved by examining the vertices of an orthogonal projection of the feasible region D into a two-dimensional

polytope in the space of the two variables used in the rewritten objective function. Pardalos (1990) proposed an algorithm that enumerates all vertices of the two-dimensional polytope until an optimal vertex is found. The algorithm may require an exponential number of steps, but its average computational time complexity is bounded by a polynomial.

2.2.2. Methods Based on Searching the Outcome Set

The objective function of problem (LMP2) can be expressed as the composite $\psi(\varphi)$ of two mappings, where, for each $x \in \boldsymbol{R}^n$, $\varphi(x) = \left(\left\langle c^1, x\right\rangle + d_1, \left\langle c^2, x\right\rangle + d_2\right)$, and, for each $y \in \boldsymbol{R}^2$, $\psi(y) = y_1 \cdot y_2$. The mapping φ maps each point $x \in D$ into a point $y = (y_1, y_2)$ where $y_1 := \left\langle c^1, x\right\rangle + d_1$ and $y_2 := \left\langle c^2, x\right\rangle + d_2$. Since y_1 and y_2 are linear functions, φ is a linear transformation and hence the linear structure of D is preserved (Rockafellar 1970). The image of D under φ is then the compact, convex polyhedron

$$Y := \left\{ y \in \boldsymbol{R}^2 \middle| y_1 = \left\langle c^1, x\right\rangle + d_1,\ y_1 = \left\langle c^2, x\right\rangle + d_2 \text{ for some } x \in D \right\}$$

called the outcome polyhedron. A global optimal solution of problem (LMP2) can be found by finding a point of Y that globally minimizes the product $y_1 \cdot y_2$. Since the search is conducted in $Y \subseteq \boldsymbol{R}^2$ rather than $\boldsymbol{R}^n$, it may be possible to economize on the computational effort required to solve problem (LMP2).

Three articles, Aneja, Aggarwal, and Nair (1984), Falk and Palocsay (1994), and Thoai (1991) proposed algorithms for solving problem (LMP2) based on searching the outcome set using outer approximation techniques. Outer approximation is a global optimization technique that uses a decreasing sequence of simple sets to approximate the

feasible region. The approximations are used in a series of optimization problems that are easier to solve than the original problem. These optimization problems are sequentially solved until a global optimal solution to the original problem is found. The technique has been very useful in solving global optimization problems in which the feasible region Z is a polytope and the global optimal solution is known to be an extreme point of Z. In this form of outer approximation, the algorithm begins by finding a simple polytope $P_0 \supset Z$ with an easily defined inequality representation and an easily calculated set of vertices. A series of algorithmic iterations follows that builds a sequence of decreasing polytopes $P_0 \supset P_1 \supset \cdots \supset Z$ in which one polytope is generated in each iteration. In an iteration k of the algorithm, the original objective function is evaluated at the extreme points of P_k to find an optimal solution v^k. If v^k is an extreme point of Z, then v^k is a global optimal solution to the original problem. Otherwise, a portion of $P_k \setminus Z$ is cut off to form P_{k+1}. The point v^k is part of the region cut off; i.e., v^k is not included in the polytope P_{k+1}. The cut is made by adding a constraint called a cutting plane constraint to the constraint set that defines P_k. The cutting plane constraint adds additional vertices to P_{k+1} that were not present in P_k and therefore they must be calculated.

Aneja, Aggarwal, and Nair (1984) proposed an algorithm that examines the solutions associated with the bicriterion programming problem:

$$\text{(BCP)} \quad \text{VMIN} \left(y_1 = \langle c^1, x \rangle + d_1, \; y_2 = \langle c^2, x \rangle + d_2 \right),$$

$$\text{s.t.} \qquad x \in D.$$

The intent of problem (BCP) is to simultaneously minimize the two criterion functions y_1 and y_2. Conflicts usually exist between the two criterion functions that prevent a single point of D from simultaneously minimizing both functions. The usual notion of an optimal solution used in single objective linear programming is replaced by the concept of efficient solutions when discussing the solutions of problem (BCP). A solution $\bar{x}$ is an efficient solution of problem (BCP) if $\bar{x} \in D$ and, whenever for each $i = 1, 2$, $\langle c^i, x \rangle + d_i \leq \langle c^i, \bar{x} \rangle + d_i$ for some $x \in D$, then $\langle c^i, x \rangle + d_i = \langle c^i, \bar{x} \rangle + d_i$, $i = 1, 2$. The set of efficient points of D is mapped by φ into a set of points on the surface of Y called the efficient frontier.

Aneja, Aggarwal, and Nair (1984) showed that a global optimal solution of problem (LMP2) is attained at an efficient extreme point x^* of D that is mapped by φ into an extreme point (y_1^*, y_2^*) in the efficient frontier of Y. Their algorithm searches the efficient frontier for an extreme point that minimizes $y_1 \cdot y_2$ by using a modified outer approximation technique. Initially the legs of a right-angle triangle form the first approximation of the efficient frontier. The "rise" and the "run" values of the slope of the hypotenuse are two positive scalar values. The functions $y_1 = \langle c^1, x \rangle + d_1$ and $y_2 = \langle c^2, x \rangle + d_2$ are multiplied by these values and then summed to form a single linear objective function. This objective function is then minimized over the feasible region D. It is well known that the minimizer $\hat{x}$ of such a linear program is an efficient extreme point of D (Steuer 1986). The solution to the linear program finds another point $(\hat{y}_1, \hat{y}_2)$ on the efficient frontier that is used to subdivide the initial triangle into two triangles. The

algorithm is then repeated using each of the smaller triangles. The algorithm terminates when there are no more extreme points of the efficient frontier that need to be searched.

In the algorithm of Aneja, Aggarwal, and Nair (1984), a new vertex must be calculated for each triangle. This is easily done by solving two systems of two equations in the unknowns y_1 and y_2. This special technique however, can not be easily extended to handle cases where $p > 2$.

Falk and Palocsay (1994) also proposed a solution algorithm that searches among the extreme points of Y using a modified outer approximation technique. In the first phase of the algorithm, the two linear programs

$$l_1 = \min_{x \in D} \langle c^1, x \rangle + d_1 \text{ and } l_2 = \min_{x \in D} \langle c^2, x \rangle + d_2$$

are solved for optimal solutions x^1 and x^2 respectively. Two initial vertices y^1 and y^2 of Y are then

$$y^1 = \left(\langle c^1, x^1 \rangle + d_1, \langle c^2, x^1 \rangle + d_2 \right) \text{ and } y^2 = \left(\langle c^1, x^2 \rangle + d_1, \langle c^2, x^2 \rangle + d_2 \right).$$

An initial polytope in outcome-space containing an optimal solution for the problem

$$\text{(YP)} \qquad \min_{y \in Y} \prod_{j=1}^{2} y_j$$

is $y_1 \geq l_1$ and $y_2 \geq l_2$ and an inequality $a_1 y_1 + a_2 y_2 \leq 1$, where a_1 and a_2 are determined such that $a_1 y_1 + a_2 y_2 = 1$ passes through the point

$$y^k = \underset{i=1,2}{\operatorname{argmin}} \left\{ y^i \middle| \langle y_1^1, y_2^1 \rangle, \langle y_1^2, y_2^2 \rangle \right\}.$$

In each iteration of the algorithm, values for a_1 and a_2 are updated and a linear program of the form

(YLP) $\min_{y \in Y} a_1 y_1 + a_2 y_2$

is solved to remove portions of the initial polytope from the search for an optimal solution for problem (YP). The new vertices generated at each iteration are easily calculated since the isovalue contours of problem (YLP) are linear. The algorithm terminates when the optimal value of problem (YLP) is one.

The algorithm proposed by Thoai (1991) for solving problem (LMP2) uses an outer approximation technique that begins by enclosing the outcome set Y in a rectangle P_0. In an iteration k of the algorithm, the extreme point $(\hat{v}_1, \hat{v}_2)$ of the outer approximation that yields the lowest value of the product $y_1 \cdot y_2$ is found. A linear program is then used to determine if the extreme point $(\hat{v}_1, \hat{v}_2)$ maps to a feasible point $\hat{x}$ of D. If not, information is obtained from the linear program to generate a cutting plane constraint that slices off the extreme point $(\hat{v}_1, \hat{v}_2)$ from the polytope P_k. The new vertices generated by the cut are then calculated using a conventional approach (see Horst, Pardalos, and Thoai 1995 or Horst and Tuy 1993). Since the method of determining these new vertices is not dependent on the fact that the dimension of the outcome set is two, Thoai's algorithm can be extended to handle cases where $p > 2$.

In the algorithms of Aneja, Aggarwal, and Nair (1984) and Thoai (1991), the only variations in the linear programs used in successive iterations involve changes in objective function coefficients. The authors gain some computational efficiency by restarting the simplex method at the optimal solution of the previous iteration. Only a few simplex pivots are then generally needed to produce a new optimal solution.

2.2.3. Methods Based on Solving a Parametric Master Problem

The difficulty in solving problem (LMP2) is caused by the product form of the objective function. Konno and Kuno (1992) added a parameter ξ and formed the following problem that they called the master problem:

$$\text{(MP2)} \quad \min F(x;\xi) = \xi\left(\langle c^1, x\rangle + d_1\right) + \frac{1}{\xi}\left(\langle c^2, x\rangle + d_2\right),$$

$$\text{s.t.} \qquad x \in D,\ \xi \geq 0.$$

Notice that for a fixed value ξ' of ξ, problem (MP2) is a linear programming problem. To solve problem (MP2), Konno and Kuno proposed using a parametric objective function simplex method to the find critical values of ξ at which new bases become optimal. The values of the objective function F are then evaluated at these bases. A global optimal solution (x^*, ξ^*) of problem (MP2) is found by choosing the basis that minimizes F over these values. Konno and Kuno (1992) showed that if (x^*, ξ^*) is an optimal solution of problem (MP2), then x^* is a global optimal solution of problem (LMP2).

Konno and Kuno tested this algorithm on randomly generated problems (LMP2) with nonnegative problem data that ranged in size from (m, n) = (30, 50) to (220, 200). Their computational experiments showed that the amount of computational time needed to solve problem (LMP2) is not much different from that required to solve linear programs of the same size.

In Konno and Kuno (1995) the authors slightly simplified the above parametric method by redefining the auxiliary parameter so that convex combinations of the two linear functions are used in the objective function of problem (MP2). This modification

makes it easier to find critical parameter values, since the interval $[0, 1]$ over which the auxiliary parameter ranges is bounded. The rest of the method remained the same.

Although Konno and Kuno (1992) did not explicitly say it, their algorithm can be viewed as searching the efficient extreme points of problem (BCP) for one that is a global optimal solution of problem (LMP2). Notice that for a sufficiently small value ξ', an extreme point optimal solution (x', ξ') to problem (MP2) coincides with an optimal solution x' of the linear program $\min\{\langle c^2, x\rangle + d_2 | x \in D\}$. Similarly, for a sufficiently large value ξ'', an extreme point optimal solution (x'', ξ'') coincides with an optimal solution x'' of the linear program $\min\{\langle c^1, x\rangle + d_1 | x \in D\}$. For any fixed value $\hat{\xi} > 0$, the objective function $F(x, \hat{\xi})$ is a composite objective function formed by multiplying the two linear functions by positive values and summing the result. It is well known that any extreme point minimizer of such a composite objective function over the feasible region D is an efficient extreme point of the problem (BCP) (Steuer 1986). The efficient extreme points of problem (BCP) are found by solving linear programs for parameter values between ξ' and ξ''. As Aneja, Aggarwal, and Nair (1984) have shown, the global solution lies at an efficient extreme point of D in problem (BCP).

A disadvantage of the algorithm of Konno and Kuno is that it may require many pivots to solve problem (MP2) for all possible parameter values. This will especially be true if there is a great conflict between the two linear functions of the objective function. If for example $c_2 = -c_1$, then every extreme point of D is an efficient extreme point of problem (BCP). Since the size of the set of extreme points of the polytope D grows

exponentially with D, the number of optimal solutions to problem (MP2) over the entire range of parameter values grows exponentially with D and is not bounded by a polynomial. Konno and Kuno in fact observed that the computational time increased as the number of local minima increased. An additional disadvantage of the Konno and Kuno algorithm is that many of the pivots performed will be unnecessary when they are to bases that do not improve on a previously found solution.

In another paper, Konno and Kuno (1990) added a convex function to the objective function of problem (LMP2) to obtain the problem (CLMP). With this addition, the objective function may no longer be quasiconcave and therefore, the global minimum may not necessarily be attained at an extreme point of the feasible region D.

To solve problem (CLMP), Konno and Kuno (1990) proposed an algorithm that solves a parametric master problem which, for a fixed parameter value, is a nonlinear convex programming problem. The algorithm involves solving this master problem a finite number of times, once for each of a finite number of prechosen values for the parameter. A troublesome aspect of the algorithm is that it is difficult to determine the proper parameter values to choose. The authors suggested choosing values for the parameter that are equally spaced in the interval of possible parameter values and solving the resulting master problems to determine a neighborhood containing a globally optimal solution to problem (CLMP). A local search is then done in that neighborhood for a globally optimal solution using the Karush-Kuhn-Tucker conditions. Care must be taken however, to attempt to define the spacing between the points to be small enough so that a global optimal solution is not missed.

The difficulty that Konno and Kuno (1990) encountered in their method in determining parameter values can be eliminated if we assume that the convex function g in the objective function of problem (CLMP) is a linear function. Problem (GLMP) is obtained by making this replacement. Konno, Yajima, and Matsui (1991) considered problem (GLMP), but they assumed that d_1 and d_2 are zero. To solve problem (GLMP), Konno, Yajima, and Matsui formulated the master problem

$$\text{(MP3)} \qquad \min F(x;\xi) = \langle c^0, x\rangle + \xi\langle c^2, x\rangle,$$

$$\text{s.t.} \qquad x \in D,$$

$$\langle c^1, x\rangle = \xi, \ \xi \geqq 0.$$

Notice that the parameter ξ appears in both the objective function and in a right-hand side of a constraint.

Konno, Yajima, and Matsui (1991) showed that x^* is a global solution of problem (GLMP) if (x^*, ξ^*) is an optimal solution of problem (MP3). Schaible and Sodini (1995) used problem (MP3) to show that a global optimal solution of problem (GLMP) lies on an edge of D.

Konno, Yajima, and Matsui (1991) proposed a parametric simplex algorithm that includes a right-hand side analysis and an objective function analysis to determine intervals of parameter values for which bases remain both feasible and optimal. The parametric analysis sweeps through parameter values from $\xi_{\min} = \min\{\langle c^1, x\rangle | x \in D\}$ to $\xi_{\max} = \max\{\langle c^1, x\rangle | x \in D\}$. The objective function F is then minimized over each of the intervals.

Konno, Yajima, and Matsui (1991) tested their algorithm on randomly generated problems of up to 350 constraints and 300 variables. They found that the problems can be solved in much the same computational time as that of solving linear programs of equal size.

The algorithm of Konno, Yajima, and Matsui (1991) suffers from the same disadvantages as the algorithm of Konno and Kuno (1992). In particular, its efficiency depends on the number of pivots performed to solve problem (MP3) for all possible parameter values. Also many of the pivots performed will be unnecessary when they yield bases that do not improve on a previously found solution.

Schaible and Sodini (1995) improved the algorithm of Konno, Yajima, and Matsui (1991). From a given simplex tableau for problem (MP3), Schaible and Sodini used parametric analysis to derive a formula that calculates the value of the objective function F as the constraint $\langle c^1, x \rangle = \xi'$ is set to increasing values of ξ'. As ξ' increases, parametric right-hand-side analysis calculates new values for the basic variables. Schaible and Sodini then derived some optimality conditions that detect when the parameter ξ' is set to a value such that from an optimal solution (x', ξ') of problem (MP3), one obtains a local minimum x' of problem (GLMP). By applying these optimality conditions, Schaible and Sodini were able to develop a simplex-based algorithm that solves problem (MP3) in a finite number of primal and/or dual simplex iterations.

The algorithm proposed by Schaible and Sodini (1995) has three advantages over the algorithm of Konno, Yajima, and Matsui (1991): (1) It may terminate before the maximum possible parameter value ξ_{max} has been reached. (2) It is more efficient in that

it may skip over local optimal solutions that do not improve the objective function value. (3) It can be used even when the feasible region is unbounded, and it can detect when problem (GLMP) is unbounded from below.

Muu and Tam (1992) also considered problem (CLMP), but in their work, the feasible region D is relaxed to a compact convex set. They seem to be the only researchers to have considered this generalization of problem (CLMP). The authors however tested their algorithm using a polytope for the feasible region.

Muu and Tam (1992) formulated the parametric master problem

$$\text{(MP 3')} \qquad \min F(x;\xi) = g(x) + \xi\left|\langle c^2, x\rangle + d_2\right|,$$

$$\text{s.t.} \qquad x \in D,$$

$$\langle c^1, x\rangle + d_1 = \xi,\ \xi \geq 0.$$

They proposed a branch and bound algorithm to solve problem (MP 3'). Branch and bound is a technique commonly used by algorithms in global optimization. Branching refers to the successive partitioning of the feasible region and bounding refers to the computation of lower and upper bounds on the global optimum over the partitions. Partitions of the feasible region that produce a lower bound on the objective function that exceeds the best upper bound found so far by the algorithm are eliminated from further consideration. Such partitions are said to be fathomed. A branch and bound algorithm terminates when all of the partitions have been fathomed.

In the algorithm of Muu and Tam (1992), partitions of the feasible region are constructed by restricting the value of $\langle c^1, x\rangle + d_1$ to values within an interval. The algorithm begins by finding an interval $I_0 := [\xi_1, \xi_2]$ of achievable values of $\langle c^1, x\rangle + d_1$

by solving the two convex programs $\xi_1 := \min\left\{\left\langle c^1, x\right\rangle + d_1 \middle| x \in D\right\}$ and $\xi_2 := \max\left\{\left\langle c^1, x\right\rangle + d_1 \middle| x \in D\right\}$. Optimal solutions u^0 and v^0 are then obtained for the two convex programs

$$\beta(\xi_i) := \min\left\{\xi_i\left[\left\langle c^2, x\right\rangle + d_2\right] + g(x) \middle| x \in D, \xi_1 \leqq \left[\left\langle c^1, x\right\rangle + d_1\right] \leqq \xi_2\right\}, \; i = 1, 2.$$

A lower bound $\beta(I_0)$ over the interval I_0 of the objective function F of problem (MP 3′) is found by selecting $\beta(I_0) := \min\{\beta(\xi_1), \beta(\xi_2)\}$. An upper bound α_0 on F is obtained by selecting $\alpha_0 := \min\{f(u^0), f(v^0)\}$. The interval I_0 is next bisected and the procedure repeated using the two subintervals. A subinterval that produces a lower bound that exceeds the current upper bound is eliminated from further consideration; i.e. that subinterval is considered to be fathomed. The procedure continues bisecting intervals I_k to generating a sequence of solutions $\{x^k\}_{k=1}^{\infty}$ that converge to a limit point x^* that is a global optimal solution. Computational experiments on problems up to $(m, n) = (30, 200)$ showed that the algorithm is very efficient when both vectors c and d are positive.

2.2.4. Methods Based on Polyhedral Annexation

A limitation of conventional optimization methods is that they can become trapped at a local minimum, or even a stationary point, if they are applied to a global optimization problem, e.g. see the algorithms proposed by Swarup (1966a, 1966b). The central problem of a global optimization method then is to overcome this limitation by providing a certification test for global optimality, and if a point is not globally optimal, determining how to move to a better solution. Tuy (1991) called this the subproblem of

"transcending the incumbent" where the incumbent is the best feasible solution found so far by an algorithm.

Let f be the objective function for problem (LMP2), and let $\bar{x}$ be a vertex of D that represents the incumbent solution for this problem. Then, from Tuy (1991), to transcend the incumbent, one must find a point in $x \in D$ such that $f(x) < f(\bar{x})$ or else establish that no such point exists, i.e. that $\bar{x}$ is a global optimal solution for problem (LMP2).

Let $G := \{x \in \Omega | f(x) \geqq f(\bar{x})\}$, where Ω is a convex set containing D. The problem of transcending the incumbent can then be restated as the following problem.

(GCP) Check if $D \subset G$ and if not, find a point $x \in D \setminus G$.

Problem (GCP) is known as the Geometric Complementary Problem.

Tuy (1990) developed the method of polyhedral annexation to solve problem (GCP). In polyhedral annexation a sequence of polytopes $P_1 \subset P_2 \subset \cdots \subset P_k \subset \cdots$ is built by adding a vertex to the polytope P_{k-1} of the previous iteration in such a way that a vertex of D is annexed into the new polytope P_k. The sequence $P_1 \cap D, P_2 \cap D, \ldots$ forms an expanding inner approximation of D. When a polytope $P_h \supseteq D$ is found, all of the extreme points of D have been searched and the algorithm terminates. Associated with the sequence of polytopes $P_1 \subset P_2 \subset \cdots \subset P_k \subset \cdots$ is the sequence of their polars $P_1^* \supset P_2^* \supset \cdots \supset P_k^* \supset \cdots$ where a polar E^* of a convex set E in $\boldsymbol{R}^n$ is defined as $E^* := \{y \in \boldsymbol{R}^n | \langle y, x \rangle \leqq 1 \text{ for all } x \in E\}$. A dual correspondence exists between the facets of a polytope P_k and the vertices of its polar P_k^*. The subproblem of determining the

inequality representation of P_k, after a new vertex has been added can then be solved by solving the easier problem of computing the vertices of P_k^*. The termination condition $P_h \supseteq D$ has the corresponding condition $P_h^* \subseteq D^*$. For a more detailed description of polyhedral annexation, see the chapters on inner approximation in Horst, Pardalos, and Thoai (1995) or in Horst and Tuy (1993).

Tuy and Tam (1992) proposed two algorithms that are derived using the polyhedral annexation method with a dualization and dimension reduction technique developed by Tuy (1991). Dualization refers to solving the original problem by solving the dual problem of generating a sequence of polars until a polar $P_h^* \subseteq D^*$ is found. The key to the dimension reduction technique is the introduction of a cone into problem (GCP). Tuy and Tam (1992) assumed that c^1 and c^2 are linearly independent vectors and then formed the cone $K := \left\{x \in \boldsymbol{R}^n \middle| \left\langle c^i, x \right\rangle \geqq 0,\ i = 1, 2\right\}$. Cone K is of interest since if $\bar{x} \in D$ is an incumbent solution, then, for any $\hat{x} \in (\bar{x} + K)$, $f(\hat{x}) \geqq f(\bar{x})$. In other words, cone K identifies points in $\boldsymbol{R}^n$ that can do no better than the incumbent solution $\bar{x}$. Computational effort might be saved using cone K since a part of the feasible region D can be eliminated from further consideration and the search narrowed to the remaining portion of D.

The first algorithm proposed by Tuy and Tam (1992) solves problem (LMP2) by solving problem (GCP) through the dualization process of generating a sequence of polars until a polar $P_h^* \subseteq D^*$ is found. Tuy and Tam (1992) showed that the polar K^* of cone K is explicitly given as $K^* = \left\{y \in \boldsymbol{R}^n \middle| y = -t_1 c^1 - t_2 c^2 \text{ for some } t_1 \geqq 0,\ t_2 \geqq 0\right\}$. Any

vertex $\hat{y}$ in a polar P_k^* lies in the polar cone K^*, and the multipliers $\hat{t}_1$ and $\hat{t}_2$ used to express $\hat{y}$ are unique, since c^1 and c^2 are linearly independent vectors. Polar cone K^* is used to solve the dual problem by building a collapsing sequence of polars $P_1^* \supset P_2^* \supset \cdots \supset P_k^* \supset \cdots$ with each polar being an improved approximation of D^*. The search is conducted in the two-dimensional space generated by c^1 and c^2 rather than in the original n-dimensional space. Solving the linear program

$$(\mathrm{LP}(\bar{t})) \qquad \max \left\{ -\bar{t}_1 \langle c^1, x \rangle - \bar{t}_2 \langle c^2, x \rangle \middle| x \in D \right\},$$

where $\bar{t}_1$ and $\bar{t}_2$ are the multipliers used to express some vertex $\bar{x} = -\bar{t}_1 c^1 - \bar{t}_2 c^2$ of P_h^*, tests for the termination condition $P_h^* \subseteq D^*$.

The second algorithm proposed by Tuy and Tam (1992) is motivated by the observation that for a fixed value of $t = (t_1, t_2)$, problem (LP(t)) is equivalent to the linear program

$$(\mathrm{LP}(\alpha)) \qquad \max \left\{ \langle -c^1 - \alpha(c^2 - c^1), x \rangle \middle| x \in D \right\}$$

where $\alpha = t_2/(t_2 + t_1) \in [0, 1]$. The first algorithm thus reduces to solving a sequence of linear programs (LP(α)) for different values of the parameter α. The second algorithm proposed by Tuy and Tam (1992) is to parametrically solve problem (LP(α)) for all of the critical values of α at which new bases become optimal. The objective function f of problem (LMP2) is evaluated at each basis and a global optimal solution chosen from those bases. The second algorithm of Tuy and Tam (1992) is essentially the same parametric problem (MP2) used by Konno and Kuno (1992).

Tuy and Tam (1992) ran computational experiments using both the first polyhedral annexation algorithm and the second parametric algorithm. Their results showed that for solving problem (LMP2), the parametric algorithm performed better than the polyhedral annexation algorithm. The polyhedral annexation algorithm is not as efficient because more simplex pivots were required than for the parametric algorithm. Tuy and Tam (1992) proposed an improved variant of the polyhedral annexation algorithm that reduces the number of pivots and the number of objective function evaluations. The authors observed that the improved algorithm may potentially be more useful for a problem with an objective function that is difficult to evaluate. The computational experiments run using the parametric algorithm on problems of up to $(m, n) = (30, 200)$ and positive problem data were in line with the results reported in Konno and Kuno (1992).

2.3. Extensions of Algorithms for Problem (LMP2) to Solve Problem (LMP) when $p \geq 3$

The polyhedral annexation method of Tuy and Tam (1992) and the outcome-space algorithms of Thoai (1991) and Falk and Palocsay (1994) can be extended to the more general problem (LMP) where $p \geq 3$. Although the algorithms remain unchanged, the subproblem of determining the new vertices becomes more difficult as the number of function terms in the objective function increases.

2.4. Methods to Solve Problems (CMP), (GCMP), and (CCMP)

Relatively little work has been done in designing exact global solution algorithms that address problems (CMP), (GCMP), and (CCMP). The algorithms that have been

proposed fall into two categories: (1) methods based on solving a reformulated problem and (2) a method based on outer approximation.

2.4.1. Methods Based on Solving a Reformulated Problem

Konno and Kuno (1992) introduced problem (CMP) where $p = 2$ and formulated a master problem by introducing a parameter into the original problem to separate the two functions of the objective function into a summation. This technique of embedding the original problem into a problem in a higher dimensional space is similar to the one used by the authors in the same paper to solve problem (LMP2). At the time, Konno and Kuno were not able to give an algorithm for solving the master problem. In Kuno and Konno (1991) the authors proposed a branch and bound algorithm along with an underestimation function to solve it. Computational results for problems of up to $(m, n) = (200, 180)$ indicated that the algorithm is efficient when the objective function is the product of a linear function and a quadratic function and the feasible region is a polytope.

Kuno, Yajima, and Konno (1993) extended the paramaterization technique of Kuno and Konno (1991) for problem (CMP) to handle cases where $p \geqq 2$. They showed that a global optimal solution to problem (CMP) can be obtained by solving the equivalent problem

$$\text{(MP4)} \qquad \min_{\xi \in \Xi} \left\{ \min_{x \in X} G(x;\xi) = \sum_{j=1}^{p} \xi_j f_j(x) \right\},$$

where $\Xi = \left\{ \xi \in R^p \,\middle|\, \prod_{j=1}^{p} \xi_j \geqq 1, \xi \geqq 0 \right\}$. For a fixed $\bar{\xi} \in \Xi$, let $x^*(\bar{\xi})$ denote an optimal solution of $\min_{x \in X} G(x;\bar{\xi}) = \sum_{j=1}^{p} \bar{\xi}_j f_j(x)$. Let $h : \Xi \to R$ be defined by $h(\xi) := G(x^*(\xi);\xi)$ for

any $\xi \in \Xi$. Solving problem (MP4) then reduces to solving the problem in R^p given by

$$(\text{MP4}') \qquad \min_{\xi \in \Xi} h(\xi).$$

Kuno, Yajima, and Konno (1993) showed that h is a concave function over Ξ and therefore a global optimal solution of problem (MP4′) exists on the boundary of Ξ. They proposed an outer approximation method for solving problem (MP4′) and tested their algorithm against two subclasses of problem (CMP): (1) problem (LMP) and (2) problems similar those tested in Kuno and Konno (1991) in which the objective function is the product of a linear and a quadratic function and the constraints are linear inequalities. Computational experiments showed that the total computational time is dominated by that needed for solving the convex minimization master problems for each parameter value. The results also showed that the number of cuts and vertices generated increases rapidly as p increased from 2 to 5. The authors asserted that this was due to inefficiencies in computing new vertices, especially when p exceeds 5. However, if p is held constant, these numbers increased very slowly as the number of constraints and variables increased. The authors concluded that their algorithm is reasonably efficient when p is less than 4.

Jaumard, Meyer, and Tuy (1997) added a convex function to the objective function of problem (CMP) to form problem (CCMP). The authors showed that problem (CCMP) can be reduced to a quasiconcave minimization problem in R^p that is a generalization of problem (MP4′) used by Kuno, Yajima, and Konno (1993). In the special case where $f_0 = 0$ in problem (CCMP), the reduced quasiconcave minimization problem in Jaumard, Meyer, and Tuy (1997) can be shown to be equivalent to the one

used by Kuno, Yajima, and Konno (1993). Jaumard, Meyer, and Tuy (1997) find a global solution of problem (CCMP) by finding an optimal solution to the quasiconcave minimization problem in $\boldsymbol{R}^p$ using a conical branch and bound method. They ran computational experiments using their algorithm on test problems similar to those used by Kuno, Yajima, and Konno (1993) and Thoai (1991). The authors report that their results are very sensitive to the magnitude of p and not as sensitive to the size (m, n) of the constraint matrix.

Sniedovich and Findlay (1995) analyzed problem (CMP) from the perspective of c-programming but did not give a complete algorithm for solving it. C-programming is a technique developed by Sniedovich (1984) for solving an optimization problem of the form

$$\text{(CP)} \qquad q := \min_{x \in X} \psi(\varphi(x)),$$

where X is some nonempty set, φ is a mapping on X with values in $\boldsymbol{R}^p$, and ψ is a differentiable and pseudo-concave function on some open set containing the set $\varphi(X) := \{\varphi(x) \mid x \in X\}$. The heart of the technique is to linearize the function ψ and transform the original optimization problem into the parametric programming problem

$$\text{(MP5)} \qquad q(\xi) := \min_{x \in X} \sum_{i=1}^{p} \xi_i \varphi_i(x), \ \xi \in \boldsymbol{R}^p.$$

Sniedovich showed that if x^* is a globally optimal solution for problem (CP), then an optimal solution ξ^* for problem (MP5) is $\xi^* = \nabla\psi(\varphi(x^*))$, where $\nabla\psi(\cdot)$ is the gradient of ψ.

For problem (CMP), the objective function can be expressed as the composite $\psi(\varphi)$ of two functions, where, for each $x \in R^n$, $\varphi(x) = (f_1(x), f_2(x), \ldots, f_p(x))$, and, for each $y \in R^p$, $\psi(y) = \prod_{i=1}^{p} y_i$. Sniedovich and Findlay claimed without proof that ψ is a differentiable and pseudo-concave function on the open convex set $\{\xi \in R^p \mid \xi > 0\}$. Since problem (CMP) satisfies the requirements of c-programming, it can be solved by solving the parametric problem

$$(\text{MP}5') \qquad q(\xi) := \min_{x \in X} \sum_{i=1}^{p} \xi_i f_i(x), \ \xi \in \Xi,$$

where Ξ is any subset of R^p such that $\nabla\psi(\varphi(x)) \in \Xi$ for all $x \in X$. In problem (MP5′), the parameter ξ appears only in the objective function, whereas in problem (MP4) the parameter ξ appears in both the objective function and in the constraints. Standard Lagrangian methods can be employed to solve problem (MP5′) for all $\xi \in \Xi$, while specialized methods are required to optimize the objective function of problem (MP4) with respect to the original variable x and the parameter ξ.

Kuno and Konno (1991) and Konno, Kuno, and Yajima (1994) considered problem (GCMP) for cases where $q = 1$ and $q \geq 1$ respectively. For $q = 1$, the master problem and solution algorithm are similar to the one used by Kuno and Konno (1991) to solve problem (CMP) when $p = 2$. Computational experiments showed that the underestimation function does not perform as well as it does for problem (CMP).

For $q \geq 1$, the master problem in Konno, Kuno, and Yajima (1994) is formulated by introducing a pair of parameters for each pair of convex functions that appear in the

objective function of problem (GCMP). The master problem is a convex minimization problem in the space R^{n+2q} and is solved using an outer approximation algorithm. Computational experiments conducted using a polyhedron for the feasible region showed that for $q = 1$, this algorithm required less than half the computational time required by the branch and bound with underestimation function algorithm proposed in Konno and Kuno (1992) to solve problem (CMP).

Tuy (1992) gave problem (CMP) as an example of an optimization problem that can be formulated as a Geometric Complementary Problem and solved it using a parametric programming problem. The parametric programming problem is a convex minimization problem in which a positive parameter vector is used to build a composite objective function from the convex functions in the objective function of problem (CMP). A complete algorithm that includes solving the parametric program was not given.

2.4.2. A Method Based on Outer Approximation

Thoai (1991) extended the algorithm based on the outer approximation technique that he proposed for solving problem (LMP2) to address the solution of problem (CMP) when $p = 2$. The main idea is to build a sequence of decreasing polytopes $P_0 \supset P_1 \supset \cdots \supset X$ of the convex feasible region X and a sequence of decreasing polytopes $S_0 \supset S_1 \supset \cdots \supset \tilde{Y}$ of the outcome set $\tilde{Y}$, where

$$\tilde{Y} = \left\{ y \in R^2 \middle| y_1 = f_1(x),\ y_2 = f_2(x) \text{ for some } x \in X \right\}.$$

Problem (CMP) is then solved by applying a modified version of the algorithm for problem (LMP2). In any iteration k, up to two cuts are introduced, one for P_k and one for S_k, to obtain tighter approximating sets.

Since the algorithm does not depend on the actual value of p, it can be extended to handle cases where $p \geqq 3$.

2.5. Methods to Solve Problem (LMP) as a Concave Minimization Problem

Konno and Kuno (1992) showed that the objective function of problem (LMP) is not a convex function over the feasible set D. Therefore, problem (LMP) is not a convex programming problem. However, since the natural logarithm function ln is a strictly increasing concave function on $(0, \infty)$, it is easy to show that the function

$$F(x) = \ln\left[\prod_{i=1}^{p}\left\langle c^i, x\right\rangle + d_i\right] = \sum_{i=1}^{p}\ln\left[\left\langle c^i, x\right\rangle + d_i\right]$$

defined for all $x \in D$ is a concave function. In addition, the optimal solution set of the concave minimization problem

$$\text{(CMIN)} \qquad \min F(x), \text{ s.t. } x \in D,$$

is identical to the optimal solution set of problem (LMP). Therefore, any concave minimization method may be applied to problem (LMP) if the objective function is replaced by its logarithmic equivalent.

Using the above transformation modification, Tuy (1991) showed that problem (LMP) could be solved in a reduced dimension space using polyhedral annexation and the dualization and dimension reduction technique. The algorithm presented in Tuy and Tam (1992) is essentially an improvement of the one in Tuy (1991).

Ryoo and Sahinidis (1996) also converted problem (LMP) into the problem (CMIN). To solve problem (CMIN), they employed a branch and bound algorithm that incorporates the use of valid inequalities to accelerate convergence. Branch and bound

algorithms may slowly converge to an optimal solution when the gap between the initial upper and lower bounds is large. A valid inequality is a inequality constraint that does not exclude any solution that yields an objective function value lower than the current best upper bound. By introducing valid inequalities into the constraint set, inferior parts of the feasible region may be removed from further consideration without eliminating possible global optimal solutions. A second use of valid inequalities is to reduce the range of values that the variables in the problem can assume. Ryoo and Sahinidis referred to these two uses of valid inequalities as range reduction mechanisms. The performance of the bounding procedure in the branch and bound algorithm is improved by using these range reduction mechanisms, since smaller-sized partitions of the feasible region are used and the variables are restricted to reduced ranges of values.

Ryoo and Sahinidis implemented the branch and bound algorithm along with the range reduction mechanisms in a computer program called BARON (Branch-And-Reduce Optimization Navigator). To more easily calculate lower bounds on the objective function F of problem (CMIN) over a partition of the feasible region, the authors replaced F by a linear underestimating function. Lower bounds were then calculated by solving linear programs. The authors tested randomly-generated problems in sizes from $(m, n) = (50, 50)$ to $(200, 200)$, with p ranging from 2 to 5. They reported that only a small fraction of the total CPU time is consumed in the range reduction mechanisms and that there seemed to be a low-order polynomial relationship between the CPU time and the value of p.

CHAPTER 3
CONCAVE MULTIPLICATIVE PROGRAMMING PROBLEMS: ANALYSIS AND AN EFFICIENT POINT SEARCH HEURISTIC FOR THE LINEAR CASE

3.1. Introduction

An important, but little researched area that deserves more attention, is the development of heuristic algorithms for finding a good solution for multiplicative programming problems. In some applications, a good, though not necessarily globally optimal solution, may adequately meet the requirements of a user (Konno and Inori 1989). In these cases, since multiplicative programming problems are known to be NP-hard, the expenditure of computational effort required to globally solve them may not be needed.

This chapter has two purposes. The first is to present an analysis of problem (P_X) when problem (P_X) is a concave multiplicative programming problem. The second purpose is to propose a heuristic algorithm designed for the case where problem (P_X) is a linear multiplicative programming problem.

The analysis of the concave multiplicative programming problem is presented in Section 3.2. This analysis shows a new way to write a concave multiplicative programming problem as a concave minimization problem and some theoretical consequences of this. It also shows some relationships between concave multiplicative programs and certain multiple-objective mathematical programs. In Section 3.3, by using some of the results of Section 3.2, we present and explain the workings of an efficient-

point search heuristic algorithm that we have developed for the linear multiplicative programming problem. Section 3.4 reports and analyzes some statistics summarizing the computational results that we obtained by coding the heuristic algorithm and applying it to 260 randomly-generated linear multiplicative programs. In Section 3.4 we also report the results of applying the heuristic algorithm to a multiplicative programming problem formed from a decision situation using real data. In Section 3.5, we discuss the major results of this chapter.

3.2. Analysis

Assume in problem (P_X) that X is a convex set and that, for each $j = 1, 2, \ldots, p$, $f_j : X \to \boldsymbol{R}$ is a concave function; i.e., assume that problem (P_X) is a concave multiplicative programming problem. Consider the function $\hat{g} : X \to \boldsymbol{R}$ defined for each $x \in X$ by

$$\hat{g}(x) = \log g(x).$$

Then, it is a simple matter to show that $\hat{g} : X \to \boldsymbol{R}$ is a concave function and that the optimal solution set to the concave minimization problem

$$\min \hat{g}(x), \text{ s.t. } x \in X, \tag{3.1}$$

is identical to the optimal solution set of problem (P_X). Thus, any concave multiplicative programming problem of the form of problem (P_X), if rewritten in the form (3.1), can be solved by applying any appropriate general-purpose concave minimization algorithm to (3.1). For discussions and reviews of concave minimization algorithms, see, for instance, Benson (1995), Benson (1996), Horst and Tuy (1993), and Pardalos and Rosen (1987).

It is interesting and useful in both practice and theory to observe that, in addition to (3.1), there is at least one other way to rewrite a concave multiplicative programming problem as a concave minimization problem. To show how this can be accomplished, we will first prove the following preliminary result.

Lemma 3.2.1. Let $a \in \boldsymbol{R}^p$ satisfy $a > 0$, and consider the nonlinear programming problem

$$v = \min \langle a, \lambda \rangle, \text{ s.t. } \lambda \in \Lambda, \tag{3.2}$$

where $\Lambda = \left\{ \lambda \in \boldsymbol{R}^p \middle| \prod_{j=1}^{p} \lambda_j \geqq 1, \lambda \geqq 0 \right\}$. Then, v is finite and problem (3.2) has at least one optimal solution.

Proof. Notice that, if $\lambda \in \Lambda$, then $\lambda > 0$ and $\langle a, \lambda \rangle > 0$. Therefore, $v > 0$. This, combined with the fact that $\Lambda \neq \emptyset$, implies that v is finite.

Now, suppose that, for each $j = 1, 2, \ldots$, there exists a vector $\lambda^j \in \Lambda$ such that

$$\langle a, \lambda^j \rangle \leqq v + \varepsilon_j,$$

where $\{\varepsilon_j\}_{j=1}^{\infty}$ is a strictly decreasing sequence of positive real numbers such that $\lim_{j \to \infty} \varepsilon_j = 0$. Then the sequence $\{\lambda^j\}_{j=1}^{\infty}$ is either bounded or unbounded.

Case 1: $\{\lambda^j\}_{j=1}^{\infty}$ is bounded. Then, for some bounded set $\hat{\Lambda} \subseteq \Lambda$, $\lambda^j \in \hat{\Lambda}$ for each $j = 1, 2, \ldots$. Therefore, by passing to an appropriate subsequence $\{\lambda^j\}_{j \in J}$ of $\{\lambda^j\}_{j=1}^{\infty}$, if necessary, we can guarantee that $\bar{\lambda} = \lim_{j \in J} \lambda^j$ exists. Furthermore, since $\lambda^j \in \hat{\Lambda} \subseteq \Lambda$ for each $j \in J$, and Λ is a closed set, $\bar{\lambda}$ belongs to Λ. By assumption,

$$\langle a, \lambda^j \rangle \leqq v + \varepsilon_j, \tag{3.3}$$

for each $j \in J$. By taking the limits over $j \in J$ on both sides of (3.3), we conclude that $\langle a, \bar{\lambda} \rangle \leqq v$. Since $\bar{\lambda} \in \Lambda$, this implies that $\bar{\lambda}$ is an optimal solution to (3.2).

Case 2: $\{\lambda^j\}_{j=1}^{\infty}$ is unbounded. Then, for some subsequence $\{\lambda^j\}_{j \in J}$ of $\{\lambda^j\}_{j=1}^{\infty}$, and for some $k \in \{1, 2, \ldots, p\}$, $\lim_{j \to \infty} \lambda_k^j = +\infty$. For each $j \in J$, since $\lambda^j \in \Lambda$, $\lambda^j > 0$.

Combined with the fact that $a > 0$, implies that, for each $j \in J$,

$$0 < a_k \lambda_k^j < \langle a, \lambda^j \rangle. \tag{3.4}$$

By assumption, for each $j \in J$,

$$\langle a, \lambda^j \rangle \leqq v + \varepsilon_j. \tag{3.5}$$

From (3.4) and (3.5), we obtain

$$a_k \lambda_k^j < v + \varepsilon_j, \tag{3.6}$$

for each $j \in J$. By taking the limits over $j \in J$ on both sides of (3.6), we conclude that $+\infty = v$, which is a contradiction. Therefore, this case cannot hold, and the proof is complete. □

Using Lemma 3.2.1, we may now establish the following theorem.

Theorem 3.2.1. Assume in problem (P_x) that X is a convex set and that $f_j : X \to \boldsymbol{R}$, $j = 1, 2, \ldots, p$, are concave functions. Let $\bar{g} : X \to \boldsymbol{R}$ be defined for each $x \in X$ by

$$\bar{g}(x) = p\left[\prod_{j=1}^{p} f_j(x)\right]^{1/p}.$$

Then $\bar{g} : X \to \boldsymbol{R}$ is a concave function.

Proof. Consider the function $h: X \to \boldsymbol{R}$ defined for each $x \in X$ by

$$h(x) = \min \sum_{j=1}^{p} \lambda_j f_j(x), \text{ s.t. } \lambda \in \Lambda, \tag{3.7}$$

where Λ is as defined in Lemma 3.2.1. From Lemma 3.2.1, since f_j is strictly positive on X for each $j = 1, 2, \ldots, p$, it follows that the minimum in (3.7) exists and is finite for each $x \in X$. If, for each $\lambda \in \Lambda$, we define a function $h_\lambda : X \to \boldsymbol{R}$ by

$$h_\lambda(x) = \sum_{j=1}^{p} \lambda_j f_j(x),$$

then for each $x \in X$, $h(x)$ may also be written as

$$h(x) = \min_{\lambda \in \Lambda} h_\lambda(x). \tag{3.8}$$

Notice that, for each $\lambda \in \Lambda$, $h_\lambda : X \to \boldsymbol{R}$ is a concave function. From this and (3.8), we conclude that $h: X \to \boldsymbol{R}$ is also a concave function (Rockafellar 1970).

To complete the proof, we will show that, for each $x \in X$, $h(x) = \overline{g}(x)$. Toward this end, fix $x \in X$, and let $\lambda(x) \in X$ denote an optimal solution to problem (3.7). From the Karush-Kuhn-Tucker necessary conditions for this problem (Bazaraa, Sherali, and Shetty 1993), since $\lambda(x) > 0$, it follows that there exists a nonnegative constant $\theta(x)$ such that

$$f_j(x) - \theta(x)\left[\prod_{k=1}^{p} \lambda_k(x)\right] \Big/ \lambda_j(x) = 0, \quad j = 1, 2, \ldots, p. \tag{3.9}$$

Since $\lambda(x) \in X$ is an optimal solution to problem (3.7), it is easy to see that

$$\prod_{k=1}^{p} \lambda_k(x) = 1.$$

Together with (3.9), this implies that

$$\lambda_j(x) f_j(x) = \theta(x), \quad j = 1, 2, \ldots, p. \tag{3.10}$$

From (3.10), it follows that

$$\lambda_j(x) = \theta(x) / f_j(x), \quad j = 1, 2, \ldots, p.$$

By substitution in

$$\prod_{j=1}^{p} \lambda_j(x) = 1,$$

this implies that

$$\theta(x) = \left[\prod_{j=1}^{p} f_j(x) \right]^{1/p}. \tag{3.11}$$

From equations (3.10) and (3.11), we see that

$$\sum_{j=1}^{p} \lambda_j(x) f_j(x) = p \left[\prod_{j=1}^{p} f_j(x) \right]^{1/p}. \tag{3.12}$$

Since $x \in X$ and $\lambda(x) \in X$ is an optimal solution to (3.7), the left-hand-side of equation (3.12) coincides with $h(x)$. By definition of $\overline{g}$, the right-hand-side of equation (3.12) equals $\overline{g}(x)$, so that the proof is complete. □

Theorem 3.2.1 can also be proven by using a composite function approach and showing several preliminary results (Avriel, Diewert, Schaible, and Zang 1987). We offer the proof here, because it is more direct and because we will use it below to help derive a corollary of interest.

Notice from Theorem 3.2.1 that, when problem (P_X) is a concave multiplicative program, the optimal solution set of problem (P_X) is identical to the optimal solution set of the concave minimization problem

$$\min \bar{g}(x), \text{ s.t. } x \in X, \tag{3.13}$$

where $\bar{g} : X \to \boldsymbol{R}$ is defined for each $x \in X$ by

$$\bar{g}(x) = p[g(x)]^{1/p}.$$

In practice, this implies that any concave multiplicative program (P_X), if rewritten in the form (3.13), can be solved by applying any suitable concave minimization algorithm to (3.13). Notice also that problem (3.13) is a simpler reformulation of problem (P_X) for the concave case than the typical reformulation used in the literature to solve problem (P_X) in the convex case (see e.g., Konno and Kuno 1992, Kuno and Konno 1991, Thoai 1991, and Kuno, Yajima, and Konno 1993).

Theorem 3.2.1 also has some interesting theoretical implications concerning the product of functions. For instance, for any finite set of concave functions f_j, $j = 1, 2, \ldots, p$, each defined on a common nonempty convex domain $X \subseteq \boldsymbol{R}^n$ and each strictly positive on this domain, it is known that the function $g : X \to \boldsymbol{R}$ defined by their product is not necessarily concave, convex, or quasiconvex on X (Kuno, Yajima and Konno 1993 and Avriel, Diewert, Schaible and Zang 1988). However, from Theorem 3.2.1, the function $f : X \to \boldsymbol{R}$ given by

$$f(x) = [g(x)]^{1/p}$$

for each $x \in X$ is a concave function on X.

In addition, Theorem 3.2.1 implies the following result concerning the product of a set of concave functions.

Corollary 3.2.1. Let X and f_j, $j = 1, 2, \ldots, p$, be defined as in Theorem 3.2.1, and suppose that $g : X \to \boldsymbol{R}$ is defined for each $x \in X$ by

$$g(x) = \prod_{j=1}^{p} f_j(x).$$

Then $g : X \to \boldsymbol{R}$ is a quasiconcave function.

Proof. Choose $\alpha \in \boldsymbol{R}$, and let

$$L_\alpha = \{x \in X \,|\, g(x) \geqq \alpha\}.$$

If $\alpha \leqq 0$, $L_\alpha = X$ is a convex set. If $\alpha > 0$, then from Theorem 3.2.1 and Rockafellar (1970), the set

$$\overline{L}_\beta = \left\{x \in X \,\middle|\, p[g(x)]^{1/p} \geqq \beta\right\}$$

is a convex set, where $\beta = p\alpha^{1/p}$. Since $\overline{L}_\beta = L_\alpha$, this implies that L_α is a convex set. Therefore, we have shown that, for any $\alpha \in \boldsymbol{R}$, L_α is a convex set. This is equivalent to showing that $g : X \to \boldsymbol{R}$ is a quasiconcave function (Bazaraa, Sherali, and Shetty 1993), so that the proof is complete. □

It follows from Corollary 3.2.1 that any concave multiplicative programming problem (P_X) is a problem involving the minimization of a quasiconcave function over a convex set. Many of the most popular algorithms for minimizing a concave function over a convex set are equally suitable for minimizing quasiconcave functions over convex sets (Horst and Tuy 1993 and Benson 1995). As a result, we see that any concave

multiplicative program (P_X) can be solved by applying any number of suitable concave minimization algorithms directly to problem (P_X). In particular, no reformulations of problem (P_X) are needed to apply these algorithms.

Remark 3.2.1. Corollary 3.2.1 has been previously shown to hold for the special case where $p=2$, X is a nonempty, compact polyhedron, and f_1 and f_2 are linear functions (see, e.g., Konno and Kuno 1992).

The next corollary of Theorem 3.2.1 concerns the minimization problem (3.7) used in the proof of the theorem. Possible uses for this corollary may include the construction of methods for finding local optimal solutions to concave multiplicative programs, although we will not investigate this here.

Corollary 3.2.2. Let X and f_j, $j=1, 2,\ldots, p$, be defined as in Theorem 3.2.1, and let Λ be defined as in Lemma 3.2.1. Then, Λ is a convex set and, for each $x\in X$, the unique optimal solution $\lambda(x)$ to problem (3.7) is given by

$$\lambda_k(x)=\left[\prod_{j=1}^{p} f_j(x)\right]^{1/p} \Big/ f_k(x),\ k=1, 2,\ldots, p.$$

Proof. Notice that Λ may be rewritten according to the relation

$$\Lambda=\left\{\lambda\in \operatorname{int} \boldsymbol{R}_+^p \,\middle|\, p\left[\prod_{j=1}^{p}\lambda_j\right]^{1/p}\geqq p\right\}, \tag{3.14}$$

where

$$\operatorname{int} \boldsymbol{R}_+^p=\left\{\lambda\in \boldsymbol{R}^p \,\middle|\, \lambda>0\right\}.$$

It is easy to see that, for each $j=1, 2,\ldots, p$, $h_j:\operatorname{int} \boldsymbol{R}_+^p\to \boldsymbol{R}$, defined for each $\lambda\in \operatorname{int} \boldsymbol{R}_+^p$ by

$$h_j(\lambda) = \lambda_j,$$

is a concave function on $\text{int } \boldsymbol{R}_+^p$ that satisfies

$$h_j(\lambda) > 0, \text{ for all } \lambda \in \text{int } \boldsymbol{R}_+^p.$$

Therefore, by Theorem 3.2.1, the function $m : \text{int } \boldsymbol{R}_+^p \to \boldsymbol{R}$ defined for each $\lambda \in \text{int } \boldsymbol{R}_+^p$ by

$$m(\lambda) = p\left[\prod_{j=1}^{p} \lambda_j\right]^{1/p},$$

is a concave function. This implies that

$$\left\{\lambda \in \text{int } \boldsymbol{R}_+^p \middle| m(\lambda) \geqq p\right\}$$

is a convex set (Rockafellar 1970). By (3.14), this proves that Λ is a convex set.

Now, fix $x \in X$, and let $\lambda(x) \in \Lambda$ denote an optimal solution to problem (3.7). From the proof of Theorem 3.2.1, this implies that, for each $k = 1, 2, \ldots, p$,

$$\lambda_k(x) = \theta(x) / f_k(x),$$

where $\theta(x)$ is given by (3.11), so that the corollary is proven. □

In addition to its relationships to concave minimization, a concave multiplicative program also has some interesting ties to multiple-objective mathematical programming. In the remainder of this section, we will show some of the theoretical relationships between concave multiplicative programs and certain multiple-objective mathematical programs. In the next section, some practical benefits of those relationships will be demonstrated.

Let $f(x)$ denote the vector

$$[f_1(x), f_2(x), \ldots, f_p(x)]^T,$$

where $f_j : X \to R$, $j = 1, 2, \ldots, p$, are the functions used in defining problem (P_X). Then, the components of the vector $f(x)$ are generally conflicting, in the sense that the infima over X of $f_j(x)$, $j = 1, 2, \ldots, p$, are generally not simultaneously achieved at the same point in X. As a result, inherent tradeoffs in the achievable values of the components of $f(x)$ over $x \in X$ are present. To account for these tradeoffs, and to seek what decision makers call a most preferred solution in situations where the goal is to attempt to simultaneously minimize $f_j(x)$, $j = 1, 2, \ldots, p$, over X, one of the most popular approaches is to consider the associated multiple-objective mathematical program

$$\text{VMIN } f(x), \text{ s.t. } x \in X. \tag{3.15}$$

In particular, in typical situations, a most preferred solution in X will exist that is also an efficient solution for (3.15), where an efficient solution is defined as follows.

Definition 3.2.1. A point $x^0 \in R^n$ is called an efficient solution for (3.15) when $x^0 \in X$ and, whenever $f(x) \leqq f(x^0)$ for some $x \in X$, then $f(x) = f(x^0)$.

An efficient solution is also called a nondominated or Pareto-optimal solution. By generating or searching the set X_E of the efficient solutions for (3.15), decision makers are able to observe the inherent tradeoffs among the objective functions f_j, $j = 1, 2, \ldots, p$, that are available over X and are often able to choose from X_E a most preferred solution. For further discussions on multiple-objective mathematical programming and its applications, the reader may consult, for instance, Cohon (1978), Evans (1984), Luc

(1989), Sawaragi, Nakayama, and Tanino (1985), Stadler (1979), Steuer (1986), Yu (1985), Zeleny (1982) and references therein.

The first relationship between multiplicative programming and multiple-objective mathematical programming is given in the following result. The proof of this result is an elementary exercise.

Proposition 3.2.1. Any optimal solution to problem (P_X) must belong to the efficient set X_E of the multiple-objective mathematical programming problem (3.15).

Notice that Proposition 3.2.1 holds for arbitrary multiplicative programming problems (P_X). The next result, however, is restricted to certain types of concave multiplicative programs.

Proposition 3.2.2. Assume in problem (P_X) that X is a compact, convex set and that $f_j : X \to \boldsymbol{R}$, $j = 1, 2, \ldots, p$, are concave functions. Then, there exists an optimal solution to problem (P_X) which is an extreme point of X.

Proof. From Theorem 3.2.1, problem (P_X) can be solved by finding an optimal solution to the concave minimization problem (3.13), where $\overline{g} : X \to \boldsymbol{R}$ is the concave function defined by

$$\overline{g}(x) = p\left[\prod_{j=1}^{p} f_j(x)\right]^{1/p},$$

for each $x \in X$. Since X is a nonempty compact, convex set, from Horst and Tuy (1993), problem (3.13) has an optimal solution that is an extreme point of X. These two observations together prove the desired result. □

Taken together, Proposition 3.2.1 and 3.2.2 imply that any concave multiplicative programming problem with a compact feasible region has at least one optimal solution that is an efficient extreme point solution to the multiple-objective mathematical programming problem (3.15). Special cases of this observations have been alluded to in the literature (see, e.g., Aneja, Aggarwal and Nair 1984 and Sniedovich and Findlay, 1995). In the next section, we put this observation to practical use.

3.3. Efficient Point Search Heuristic

Assume in this section that, in problem (P_X),

$$X = \left\{ x \in \boldsymbol{R}^n \middle| Ax \leqq b \right\}$$

is a compact polyhedron, where A is an $m \times n$ matrix and $b \in \boldsymbol{R}^m$, and that for each $j = 1, 2, \ldots, p$, $f_j(x) = \langle c^j, x \rangle$, where $c^j \in \boldsymbol{R}^n$ for each $j = 1, 2, \ldots, p$. Then problem (P_X) is a linear multiplicative programming problem or, more briefly, a linear multiplicative program (Konno and Kuno 1992). We have designed and tested a heuristic algorithm for this problem, based in part on some of the results in the previous section. In this section, we will formally state this heuristic algorithm and explain its workings.

The multiple-objective program (3.15) associated with a linear multiplicative problem may be written as

$$\text{VMIN } Cx, \text{ s.t. } Ax \leqq b, \tag{3.16}$$

where C is the $p \times n$ matrix whose jth row equals $(c^j)^T$, $j = 1, 2, \ldots, p$. Problem (3.16) is a multiple-objective linear programming problem (Steuer 1986 and Yu 1985). Let X_{ex} denote the set of extreme points of

$$X = \left\{ x \in R^n \middle| Ax \leq b \right\}.$$

Then, by Proposition 3.2.1 and 3.2.2, an optimal solution to the linear multiplicative programming problem can be found in the set

$$X_{E,ex} = \left(X_E \bigcap X_{ex} \right)$$

of efficient extreme points of problem (3.16). The set $X_{E,ex}$ is finite, and various procedures have been developed for generating it in its entirety (see, e.g., Steuer 1986, Yu 1985 and Steuer 1983).

It follows that, in theory, at least, a global optimal solution to a linear multiplicative problem can be found by completely enumerating the set $X_{E,ex}$ of efficient extreme points of the associated multiple-objective linear programming problem (3.16) and, from this set, choosing the point(s) with the smallest value of

$$g(x) = \prod_{j=1}^{p} \left\langle c^j, x \right\rangle$$

(see, e.g., Sniedovich and Findlay 1995). Unfortunately, as we shall see later, in practice the exponential growth in the size of $X_{E,ex}$ as a function of problem size (Steuer 1986) renders this approach impractical for many cases.

The approach of the heuristic algorithm is to efficiently search a dispersed, carefully chosen sample of candidate points from $X_{E,ex}$ in order to find an attractive solution to the linear multiplicative programming problem. To describe and explain the workings of the heuristic, we must first present some theoretical background from the theory of multiple-objective linear programming.

Let

$$W = \{w \in \boldsymbol{R}^p \mid \langle e, w \rangle \leqq M, w \geqq e\},$$

where $e \in \boldsymbol{R}^p$ is a vector with each entry equal to 1.0, and M is a positive real number. For sufficiently large M, from Philip (1972) it is known that a point x^0 belongs to the efficient set X_E of (3.16) if and only if x^0 is an optimal solution to the weighted-sum problem

$$\min \langle w^T C, x \rangle, \text{ s.t. } Ax \leqq b, \tag{3.17}$$

for some $w = w^0 \in W$. We will assume henceforth that M is chosen to be large enough to guarantee that this property holds. It is also well known that the efficient set X_E for (3.16) is given by

$$X_E = \bigcup \{X_w \mid w \in W\},$$

where, for each $w \in W$, X_w denotes the optimal solution set of the linear program (3.17) (Steuer 1986 and Yu 1985). Since the optimal solution set to (3.17) for any $w \in W$ is a face of

$$X = \{x \in \boldsymbol{R}^n \mid Ax \leqq b\},$$

it follows that the efficient set X_E for (3.16) is equal to the union of the faces X_w, $w \in W$, of X. Although X_E is a connected set (Yu 1985), it is generally nonconvex. The heuristic algorithm will individually identify efficient faces X_w, $w \in W$, of X, and find an approximately-optimal extreme point solution to the problem

$$\min \prod_{j=1}^{p} \langle c^j, x \rangle, \text{ s.t. } x \in X_w \tag{3.18}$$

for each efficient face X_w that it finds.

Let

$$Y = \left\{ y \in \boldsymbol{R}^p \middle| y = Cx, \text{for some } x \in X \right\},$$

$$Y^{\geq} = \left\{ y \in \boldsymbol{R}^p \middle| y \geqq \hat{y}, \text{for some } \hat{y} \in Y \right\}.$$

To aid in its search, the heuristic algorithm will solve the linear program

$$\min \left\langle w^T C, x \right\rangle \tag{3.19a}$$

$$\text{s.t. } Cx \leqq y, \tag{3.19b}$$

$$Ax \leqq b, \tag{3.19c}$$

for various values of $y \in Y^{\geq}$ and $w \in W$. The heuristic relies in part upon the properties of problem (3.19) given in the next three results. The first two results follow easily from Benson (1978).

Theorem 3.3.1. Suppose that $x^0 \in R^n$ and let $y^0 = Cx^0$. Then, x^0 is an efficient solution for (3.16) if and only if, with $y = y^0$, x^0 is an optimal solution to (3.19) for every $w \in W$.

Theorem 3.3.2. If $y \in Y^{\geq}$ and $w \in W$, then (3.19) has at least one optimal solution, and any optimal solution for (3.19) is an efficient solution for (3.16).

Theorem 3.3.3. Suppose in (3.19) that $w = w^0 \in W$ and that $y = y^0 = Cx^0$, where x^0 is an efficient solution for (3.16). Let $\left(u^{0T}, z^{0T}\right)$ denote any optimal solution to linear programming dual of (3.19), where u^0 represents the dual variables corresponding to the constraints $Cx \leqq y^0$ of (3.19). Let $\overline{w}^0 = u^0 + w^0$ and let $v_0 = \left(\overline{w}^0\right)^T Cx^0$. Then, x^0 belongs to the efficient face $X_{\overline{w}^0}$ of X, and $X_{\overline{w}^0}$ can be represented as

$$X_{\overline{w}^0} = \left\{ x \in X \middle| \left(\overline{w}^0\right)^T Cx = v_0 \right\}.$$

Proof. To prove the theorem, we will show that, with $w = \overline{w}^0$, x^0 is an optimal solution to problem (3.17). Suppose in (3.19) that $w = w^0 \in W$ and that $y = y^0 = Cx^0$, where x^0 is an efficient solution for (3.16) given in the theorem. The dual linear program to (3.19) is then given by

$$\max -\langle y^0, u \rangle - \langle b, z \rangle,$$

$$\text{s.t. } -C^T u - A^T z = C^T w^0,$$

$$u, z \geqq 0.$$

From Theorem 3.3.1, x^0 is an optimal solution to (3.19) when $w = w^0$ and $y = y^0$. By the duality theory of linear programming (Murty 1983), since $\left(u^{0T}, z^{0T}\right)$ is an optimal solution to the linear programming dual of (3.19) when $w = w^0$ and $y = y^0$, this implies that

$$\left(w^0\right)^T Cx^0 = -\langle y^0, u^0 \rangle - \langle b, z^0 \rangle.$$

By rearranging this equation and using the definitions of y^0 and $\overline{w}^0$, we obtain

$$\left(\overline{w}^0\right)^T Cx^0 = -\langle b, z^0 \rangle. \tag{3.20}$$

With $w = \overline{w}^0$, the dual linear program to (3.17) may be written as

$$\max -\langle b, z \rangle, \tag{3.21a}$$

$$\text{s.t. } -A^T z = C^T \overline{w}^0, \tag{3.21b}$$

$$z \geqq 0. \tag{3.21c}$$

Let z denote an arbitrary feasible solution to problem (3.21). From the definitions of u^0

and $\overline{w}^0$, this implies that (u^{0T}, z^T) is a feasible solution to the dual linear program of (3.19). Since (u^{0T}, z^{0T}) is an optimal solution to the latter problem, it follows that

$$-\langle y^0, u^0\rangle - \langle b, z^0\rangle \geqq -\langle y^0, u^0\rangle - \langle b, z\rangle,$$

or, equivalently,

$$-\langle b, z^0\rangle \geqq -\langle b, z\rangle.$$

Notice that, since (u^{0T}, z^{0T}) is an optimal solution to the dual linear program to (3.19), z^0 is a feasible solution to (3.21). By the choice of z, the preceding two statements imply that z^0 is an optimal solution to (3.21). Since x^0 is an efficient solution for (3.16), with $w = \overline{w}^0$, x^0 is a feasible solution for (3.17). From (3.20) and the duality theory of linear programming (Murty 1983), since z^0 is an optimal solution to (3.21), this implies that, with $w = \overline{w}^0$, x^0 is an optimal solution to (3.17), and the proof is complete. □

Notice in Theorem 3.3.3 that, for any $t > 0$, $X_{t\overline{w}^0} = X_{\overline{w}^0}$. This implies that, in Theorem 3.3.3, when $\overline{w}^0 \notin W$, there exists a $t \in (0, 1)$ such that $t\overline{w}^0 \in W$ and $X_{t\overline{w}^0} = X_{\overline{w}^0}$. Thus, in Theorem 3.3.3, when $\overline{w}^0 \notin W$, $X_{\overline{w}^0}$ has an alternate representation $X_{\overline{\overline{w}}^0}$ for which $\overline{\overline{w}}^0 \in W$. For simplicity, we may and will assume without loss of generality that in Theorem 3.3.3, $\overline{w}^0 \in W$.

To generate various points $y \in Y^{\geq}$ for use in problem (3.19), the heuristic algorithm will rely upon the two concepts defined in the next two definitions (see, e.g., Zeleny 1982).

Definition 3.3.1. The point $y^I \in \boldsymbol{R}^p$ is called the ideal point of Y when, for each $j = 1, 2, \ldots, p$, y_j^I equals the minimum value of y_j over Y.

Definition 3.3.2. The point $y^{AI} \in \boldsymbol{R}^p$ is called the anti-ideal point of Y when, for each $j = 1, 2, \ldots, p$, y_j^{AI} equals the maximum value of y_j over Y.

Notice that y_j^I and y_j^{AI} generally do not belong to Y. The algorithm uses these two points as anchor points in an initialization procedure whose goal is, in part, to generate a dispersed sample of points from $Y^{\geq}$.

The heuristic algorithm may be stated as follows.

Algorithm 3.3.1. Efficient Point Search Heuristic Algorithm

Initialization Phase. See Steps 1 through 5 below.

Step 1. Find the ideal and anti-ideal points y^I and y^{AI} of Y.

Step 2. Find an optimal solution $\left[(x^*)^T, \alpha^*\right] \in \boldsymbol{R}^{n+1}$ to the linear program

$$\max \alpha,$$

$$\text{s.t.} \quad y^{AI} + \alpha\left(y^I - y^{AI}\right) \geqq Cx,$$

$$Ax \leqq b,$$

$$\alpha \geqq 0,$$

and set $y^* = y^{AI} + \alpha^*\left(y^I - y^{AI}\right)$.

Step 3. Choose a positive integer S and, for each $i = 1, 2, \ldots, S$, let

$$y^i = y^{AI} + (i/S)\left(y^* - y^{AI}\right).$$

Step 4. Choose a positive integer N such that $1 \leqq N \leqq M - p + 1$, let $w^0 = e \in \boldsymbol{R}^p$, and, for each $j = 1, 2, \ldots, p$, define $w^j \in \boldsymbol{R}^p$ by

$$w_i^j = \begin{cases} 1, & \text{if } i \neq j, \\ N, & \text{if } i = j. \end{cases}$$

Step 5. Set $UB = +\infty$, $i = 0$ and $j = 0$.

Efficient Point Search Phase. See Steps 1 through 6 below.

Step 1. Set $y = y^j$ and $w = w^j$, and find any optimal solution $\bar{x}^{ij}$ to linear program (3.19).

Step 2. Set $y = C\bar{x}^{ij}$ and $w = w^j$ in (3.19), and compute any optimal solution $\left[\left(u^{ij}\right)^T, \left(z^{ij}\right)^T\right]$ to the dual linear program to (3.19), where u^{ij} denotes the optimal dual variables corresponding to the constraints $Cx \leqq y$ of (3.19).

Step 3. Let $\bar{w}^{ij} = u^{ij} + w^j$. If $\bar{w}^{ij}$ is a positive multiple of $\bar{w}^{i'j'}$ for some $i' \leq i$ and $j' \leq j$ such that $(i', j') \neq (i, j)$, then go to Step 6. Otherwise, continue.

Step 4. Let $v_{ij} = \left(\bar{w}^{ij}\right)^T C\bar{x}^{ij}$. For each $h = 1, 2, \ldots, n$, calculate a_h according to the formula

$$a_h = \sum_{k=1}^{p} \left[\prod_{t \neq k} \left\langle c^t, \bar{x}^{ij} \right\rangle \right] \left[c_h^k \right], \tag{3.22}$$

and find any basic optimal solution x^{ij} to the linear program

$$\min \langle a, x \rangle, \tag{3.23a}$$

$$\text{s.t. } \left(\bar{w}^{ij}\right)^T Cx = v_{ij}, \tag{3.23b}$$

$$Ax \leqq b. \tag{3.23c}$$

Step 5. If $\prod_{k=1}^{p} \left\langle c^k, x^{ij} \right\rangle \geq UB$, go to Step 6. Otherwise, set $\bar{x} = x^{ij}$, and $UB = \prod_{k=1}^{p} \left\langle c^k, \bar{x} \right\rangle$, and go to Step 6.

Step 6. Set $j = j+1$. If $j \leqq p$, go to Step 1. Otherwise, set $i = i+1$ and $j = 0$. If $i \leqq S$, go to Step 1. Otherwise, Stop: $\bar{x} \in X_{E,ex}$ is the recommended solution to the linear multiplicative programming problem.

In the initialization phase of the algorithm, samples of points from $Y^{\geq}$ and from W are generated. To generate the sample of points from $Y^{\geq}$, Step 2 of this phase determines the point y^* between y^{Al} and y^{l} such that, of all line segments with endpoints y^{Al} and y that lie in $Y^{\geq}$ and for which y lies on the line segment connecting y^{Al} and y^{l}, the line segment L connecting y^{Al} and y^* has maximum norm. The sample $\{y^i | i = 1, 2, \ldots, S\}$ of points from $Y^{\geq}$ is then generated in Step 3 of this phase by partitioning L into S line segments of equal length, where S is a positive integer chosen by the user. In Step 4, a sample of $p+1$ all-integer vectors from W is generated, where for p of these vectors, the value N of one of the components is chosen by the user from the set $\{1, 2, \ldots, M - p + 1\}$.

Each iteration of the efficient point search phase of the heuristic executes two key operations. First, it identifies an efficient face $X_{\overline{w}^{ij}}$ of X. Second, unless this face has been previously identified during an earlier execution of this phase, with $w = \overline{w}^{ij}$ in problem (3.18), by using a first-order linear approximation to the objective function of this problem, it finds an extreme point x^{ij} of X in this efficient face that is an approximate optimal solution to (3.18).

Steps 1 through 3 of the efficient point search phase of the algorithm identify an efficient face of X. In Step 1, with $y = y^i \in Y^{\geq}$ and $w = w^j \in W$, the linear program

(3.19) is solved for any optimal solution $\bar{x}^{ij}$. By Theorem 3.3.2, this optimal solution must exist and is an efficient solution for (3.16). In Steps 2 and 3, with $y = C\bar{x}^{ij}$ and $w = w^j$ in (3.19), the dual linear program to (3.19) is solved to yield the vector $u^{ij} \in \boldsymbol{R}^p$, and the weighting vector $\bar{w}^{ij} = u^{ij} + w^j$ is computed. From Theorem 3.3.3, the face $X_{\bar{w}^{ij}}$ corresponding to this weighting vector is an efficient face for (3.16) and contains $\bar{x}^{ij}$. Furthermore, from the same theorem, this face can be written as

$$X_{\bar{w}^{ij}} = \left\{ x \in \boldsymbol{R}^n \middle| Ax \leqq b, \left(\bar{w}^{ij}\right)^T Cx = v_{ij} \right\}, \tag{3.24}$$

where $v_{ij} = \left(\bar{w}^{ij}\right)^T C\bar{x}^{ij}$. Step 3 checks whether or not $X_{\bar{w}^{ij}}$ has been identified during a previous execution of this phase of the algorithm. If so, the algorithm proceeds to Step 6 to prepare for another possible iteration of the efficient point search phase of the heuristic. Otherwise, control shifts to Steps 4-5.

In Steps 4-5 of the efficient point search phase, problem (3.18) is approximately solved using a new efficient face X_w as the feasible region. In particular, in Step 4, (3.22) is first used to construct the nonconstant portion of a first-order Taylor series linear approximation $\langle a, x \rangle$ of the objective function of problem (3.18) at $x = \bar{x}^{ij} \in X_{\bar{w}^{ij}}$. Next, using the representation (3.24) of the efficient face $X_{\bar{w}^{ij}}$, an extreme point minimizer x^{ij} of $\langle a, x \rangle$ over $X_{\bar{w}^{ij}}$ is found by solving the linear program (3.23). Notice that $x^{ij} \in X_{E,ex}$ (see Rockafellar 1970). In Step 5, the value achieved by x^{ij} in the objective function of the linear multiplicative problem is compared to the smallest value *UB* found thus far for this objective function by the search. If x^{ij} achieves a smaller objective function value

than UB, x^{ij} becomes the new incumbent solution $\bar{x}$ and UB is reduced in value accordingly.

Notice that the performance of the heuristic algorithm depends in part upon the number, locations, and dimensions of the efficient faces (3.24) that are searched via problem (3.23). This, in turn, is partially dependent upon the sizes of the parameters S and N chosen by the user. The goal is to search as many points of $X_{E,ex}$ as possible by generating a variety of distinct efficient faces (3.24) of large dimensions that are dispersed widely throughout X_E. Notice that, since each efficient face identified by the heuristic is given in the form (3.24) and searched by solving linear program (3.23), the individual points in $X_{E,ex}$ that are searched by the algorithm are searched implicitly rather than explicitly, i.e., they do not need to be explicitly enumerated.

3.4. Computational Results

The heuristic algorithm described in Section 3.3 has the following attractive characteristics:

(a) it can be implemented using only linear programming methods;

(b) it generally implicitly searches many efficient extreme points of (3.16) at once by optimizing over entire efficient faces of (3.16), rather than by explicitly examining individual efficient extreme points of (3.16);

(c) it allows the user to manipulate the nature and extent of the efficient face search through the choices for the input parameters S and N;

(d) it finds efficient faces of (3.16) by attempting to globally sample from a variety of regions of the efficient set.

To evaluate the effectiveness in practice of the heuristic algorithm and its features, we have written a VS FORTRAN computer code for the algorithm and used it to solve 260 linear multiplicative programming problems of various sizes. To execute the code on these 260 problems, we used an IBM ES/9000 model 831 mainframe computer. As a further illustration of the effectiveness in practice of the heuristic algorithm, we solved a multiple-object linear programming problem in forest management that was derived from a real decision situation using real data.

To implement Step 3 of the initialization phase of the algorithm, we chose to set $S = 4$, so that a sample of five points lying between y^{AI} and y^{I} in $Y^{\geq}$ is always generated in this step. We used a value of $N = 9$ in Step 4 of the initialization phase to help generate the sample of $p+1$ points from W.

To solve the linear programming problems called for by the heuristic, the computer code uses the simplex method procedures given in the subroutines of the Optimization Subroutine Library (International Business Machines 1990). These subroutines employ anticycling rules to handle degeneracy as needed. Therefore, they are especially appropriate for solving instances of problem (3.23), since these problems always contain degenerate extreme points.

Let

$$\text{int } \boldsymbol{R}^n_+ = \left\{ x \in \boldsymbol{R}^n \middle| x > 0 \right\},$$

and suppose that k is a positive integer. To generate the 260 test problems, we used the following random procedure. First, for each $j = 1, 2, \ldots, p$, we generated the elements of the vector $c^j \in \boldsymbol{R}^n$ by randomly drawing elements from the set $\{1, 2, \ldots, 10\}$. Next, we

generated a nonempty, compact polyhedral feasible region $X \subseteq \text{int } \boldsymbol{R}^n_+$. This region can be written as

$$X = \left\{ x \in \boldsymbol{R}^n \middle| Px \geqq q, 1 \leqq x_j \leqq \hat{q},\ j = 1, 2, \ldots, n \right\},$$

where P is a $k \times n$ matrix, $q \in \boldsymbol{R}^k$, and $\hat{q} \in \boldsymbol{R}$. To accomplish this, first the elements of P were generated by randomly choosing elements from the set $\{1, 2, \ldots, 10\}$. Next, for each $i = 1, 2, \ldots, k$, the formula

$$q_i = \sum_{j=1}^{n} p_{ij}^2$$

was used to calculate q_i, and, finally, $\hat{q}$ was chosen according to the rule

$$\hat{q} = \max \left\{ q_i \middle| i = 1, 2, \ldots, k \right\}.$$

Each test problem was constructed to belong to one of four categories, where a category is defined by the number p of linear functions used in the objective function $\prod_{j=1}^{p} \left\langle c^j, x \right\rangle$ of the test problem. The values $p = 2, 3, 4, 5$ were chosen to define these categories. We chose these categories in this way because empirical evidence seems to indicate that the complexity of these problems is more sensitive to the magnitude of p than to the magnitudes of k or n (Kuno, Yajima and Konno 1993). Within each category, the test problems were classified into subcategories of 10 problems, each defined by the values of the ordered pair (k, n).

To help evaluate the attractiveness of the solutions found by the heuristic algorithm, we found a global optimal solution for each test problem by completely enumerating all of the efficient extreme points of the associated multiple-objective linear

program (3.16). To accomplish this, we use the ADBASE computer code developed by Steuer (1983).

Some statistics summarizing the results of these computations are presented in Tables 3.1-3.4. In each table, each row gives average statistics for a subcategory (k,n) of 10 problems, a measure of the worst case performance of the heuristic, and the number of problems in a category for which a global optimal solution was found. The first statistic is the average number of efficient extreme points found by ADBASE in solving the problems by complete enumeration. In some sense, the magnitudes of these numbers correspond to the average relative difficulties, by subcategory, of each group of 10 linear multiplicative programs in a subcategory. The second statistic is the average efficiency rating r given by

$$r = 1 - [(z_H - z_{\min})/(z_{\max} - z_{\min})],$$

where z_H is the objective function value returned by the heuristic, and where $z_{\min}$ and $z_{\max}$ are the global minimum and maximum values of the objective function of the test problem over the corresponding set of efficient extreme points of (3.16). Thus, $0 \leqq r \leqq 1$, and the closer r is to 1.0, the more attractive the value z_H returned by the heuristic is relative to the actual global minimum value $z_{\min}$. The third statistic given for each subcategory in these tables is the average CPU time (seconds) that the heuristic needed to solve a problem in the subcategory. The fourth statistic shows the lowest efficiency rating calculated for a problem in the subcategory. It gives a measure of the worst case performance of the heuristic algorithm when applied to the 10 problems in a subcategory.

Table 3.1. Computational Results: $p = 2$.

Subcategory k	n	Avg No. Eff. Points	Avg. Eff. Rating r	Avg. Solutions Time (sec.)	Lowest Eff. Rating r	No. Exact Solutions
25	20	28.8	1.000	0.227	1.000	10
25	30	28.8	1.000	0.241	1.000	10
30	40	47.9	1.000	0.389	1.000	10
40	30	28.2	1.000	0.328	1.000	10
40	50	47.0	0.999	0.504	0.996	8
50	40	35.1	0.999	0.453	0.999	9
50	60	29.2	1.000	0.556	1.000	10
60	70	62.3	1.000	1.070	1.000	10

The fifth statistic is the number of problems in a category for which the heuristic algorithm found a global optimal solution.

These four tables show that the solutions returned by the heuristic algorithm give, on the average, quite accurate estimates of the actual global minimum values for the 260 linear multiplicative test problems generated. This is indicated by the fact that average efficiency ratings by subcategory always were at least 0.920, and in approximately 96% of the subcategories exceeded 0.950. It is noteworthy that, for these problems, these ratings r by subcategory do not seem to decline significantly as p, k, and n increase in

Table 3.2. Computational Results: $p = 3$.

Subcategory k	n	Avg. No. Eff. Ext. Points	Avg Eff. Rating r	Avg. Solutions Time (sec.)	Lowest Eff. Rating r	No. Exact Solutions
25	20	330.6	0.985	0.321	0.951	4
25	30	896.8	0.960	0.469	0.708	5
30	40	873.3	0.987	0.543	0.884	7
40	30	949.3	0.993	0.609	0.968	6
40	50	2073.7	0.920	0.967	0.806	4
50	40	1484.9	0.993	0.908	0.961	7
50	60	2846.3	0.995	1.298	0.978	6
60	70	5867.5	0.969	2.495	0.799	2

Table 3.3. Computational Results: $p = 4$.

Subcategory k	Subcategory n	Avg. No. Eff. Ext. Points	Avg Eff. Rating r	Avg. Solutions Time (sec.)	Lowest Eff. Rating r	No. Exact Solutions
25	20	2789.5	0.998	0.426	0.993	4
25	30	7245.9	0.992	0.598	0.945	5
30	40	23656	0.986	1.019	0.947	1
40	30	19034	0.978	0.998	0.923	2
40	50	50889	0.969	1.539	0.918	0
50	40	59443	0.969	1.587	0.843	2
50	50	83780	0.981	1.901	0.890	3

value. In addition, with the exception of one subcategory, a global optimal solution was found for at least one problem in each subcategory.

The average solution times by subcategories shown in the four tables indicate that, for these test problems, the computational effort required by the heuristic was rather small. In fact, these average times were always less than 2.50 seconds. In comparison to exact algorithms that have been used in test situations to globally solve linear multiplicative problems, these times are generally either at least as small or much smaller (see, e.g., Kuno, Yajima, Konno 1993 and Ryoo and Sahinidis 1996). Furthermore, in contrast to solution times for exact algorithms, these average solution times seem much less sensitive to increases in p, n, k or to increases in the average number of efficient

Table 3.4. Computational Results: $p = 5$.

Subcategory k	Subcategory n	Avg. No. Eff. Ext. Points	Avg. Eff. Rating r	Avg. Solutions Time (sec.)	Lowest Eff. Rating r	No. Exact Solutions
10	20	1331.4	0.993	0.353	0.941	5
20	10	527.1	0.998	0.294	0.993	2
25	30	57115	0.995	0.962	0.992	2

extreme points that exist in the corresponding problems (3.16); see Kuno, Yajima, Konno (1993) and Ryoo and Sahinidis, (1996).

Finally, it is worth noting that we were able to apply the heuristic to much larger problems than those reported in Tables 3.1-3.4. However, the number of efficient extreme points in the associated multiple-objective linear programming problems (3.16) for these cases always exceeded 200,000. Since the ADBASE code cannot be used to find all of the efficient extreme points for such problems, we were unable to completely enumerate the sets of efficient extreme points to find z_{min} and r values for these problems. Thus, we are as yet not able to draw conclusions concerning the accuracy of the heuristic for any problems larger than those reported in Tables 3.1-3.4.

To further illustrate the effectiveness in practice of the heuristic algorithm, we solved a real application problem in forest management that was studied in Steuer and Schuler (1978) as a multiple-objective linear programming problem. The problem involves the allocation of land and budget monies in a way that seeks to maximize objectives in timber production, hunting and cattle grazing in the Swan Creek subunit of the Mark Twain National Forest. Steuer and Schuler (1978) provide actual data used to formulate their multiple-objective linear programming problem. The problem contains 31 decision variables, 5 linear objective functions, and 13 constraints. Our multiplicative programming problem was formed from this problem by multiplying the 5 linear objective functions together to form a single objective function. The heuristic was then used to search for an approximate solution that maximizes this single objective function subject to the constraints of the forest management multiple-objective linear programming problem.

To help evaluate the attractiveness of the solution found by the heuristic algorithm, we found a global optimal solution by enumerating the 83 efficient extreme points of the associated forest management multiple-objective linear program using the ADBASE computer code. An efficiency rating of $r = 0.999$ was calculated using the slightly modified equation

$$r = 1 - [(z_{\max} - z_H)/(z_{\max} - z_{\min})]$$

since this multiplicative programming problem is a maximization problem rather than a minimization problem. This efficiency rating indicates that the heuristic algorithm returned an attractive value z_H relative to the actual global maximum value $z_{\max}$.

3.5. Discussion

The results of this chapter imply that there are at least two ways to rewrite a concave multiplicative programming problem as a concave minimization problem. It follows that concave minimization theory and methods can be used in these ways to analyze and solve concave multiplicative programs. The results also imply that a concave multiplicative programming problem can be analyzed and solved directly, without any reformulation, as a quasiconcave minimization problem over a convex set. Furthermore, the analysis in the chapter implies that any concave multiplicative programming problem (P_X) with a compact feasible region has at least one optimal solution that is an efficient extreme point solution of the associated multiple-objective mathematical programming problem (3.15). Therefore, the opportunity exists for devising solution methods for such problems (P_X) that search among the efficient extreme points of the associated multiple-objective problems (3.15). The chapter proposes a heuristic algorithm that takes this

approach for solving linear multiplicative programs. From the computational results presented for this heuristic algorithm, we conclude that its features and performance offer significant potential for conveniently finding very attractive solutions with relatively little computational effort to the various applications using linear multiplicative programming encountered in practice. Thus, the theoretical and algorithmic results presented in this chapter offer some potential new avenues for more effectively analyzing and solving multiplicative programming problems of various types.

CHAPTER 4
A GENERAL MULTIPLICATIVE PROGRAMMING PROBLEM IN OUTCOME-SPACE

4.1. Introduction

Recall from Chapter 1 that the multiplicative programming problem is given by

$$(P_X) \qquad v_x = \min \prod_{j=1}^{p} f_j(x), \text{ s.t. } x \in X,$$

where $p \geqq 2$ is an integer, X is a nonempty set in $\boldsymbol{R}^n$, and, for each $j = 1, 2, \ldots, p$, $f_j : X \to \boldsymbol{R}$ satisfies $f_j(x) > 0$ for all $x \in X$. For simplicity, we assume that the minimum v_x in problem (P_X) is achieved.

For any $x \in \boldsymbol{R}^n$, let $f(x)$ denote the p-vector with jth entry equal to $f_j(x)$, $j = 1, 2, \ldots, p$. Let $y \in \boldsymbol{R}^p$ denote the p-vector with jth entry equal to y_j, $j = 1, 2, \ldots, p$. For each $j = 1, 2, \ldots, p$, let $\hat{y}_j \in \boldsymbol{R}$ satisfy

$$\hat{y}_j \geqq \sup f_j(x), \text{ s.t. } x \in X,$$

where $\hat{y}_j = +\infty$ is possible, and let $\hat{y} \in \boldsymbol{R}^p$ denote the vector with jth entry equal to $\hat{y}_j$, $j = 1, 2, \ldots, p$. Although various outcome-space reformulations of problem (P_X) have been proposed for solution purposes, one of the most common reformulations is given by the problem

$$(P_{Y^{\leqq}}) \qquad v_y = \min g(y), \text{ s.t. } y \in Y^{\leqq},$$

where

$$Y^{\leqq} = \left\{ y \in \boldsymbol{R}^p \middle| f(x) \leqq y \leqq \hat{y} \text{ for some } x \in X \right\}, \tag{4.1}$$

and where, for each $y \in Y^{\leqq}$, $g : Y^{\leqq} \to \boldsymbol{R}$ is defined by

$$g(y) = \prod_{j=1}^{p} y_j. \tag{4.2}$$

For example, problem $(P_{Y^{\leqq}})$ is essentially the reformulation of problem (P_X) used in the algorithms of Benson (1998c), Falk and Palocsay (1994) and Thoai (1991). Notice that since X is nonempty, $Y^{\leqq}$ is a nonempty set. By constructing appropriate global solution algorithms for problem $(P_{Y^{\leqq}})$, this problem provides us with the opportunity to solve problem (P_X) by working in the outcome-space $\boldsymbol{R}^p$ of the problem, rather than in the decision space $\boldsymbol{R}^n$, which is generally much larger than $\boldsymbol{R}^p$. In order to globally solve problem $(P_{Y^{\leqq}})$, it is important to understand the properties of the set $Y^{\leqq}$ defined by (4.1), of the function g defined by (4.2), and of problem $(P_{Y^{\leqq}})$ itself.

This chapter undertakes a mathematical analysis of the outcome-space reformulation $(P_{Y^{\leqq}})$ of problem (P_X). The analysis is organized according to whether or not the outcome-space problem satisfies conditions for the general case, the convex case, or the polyhedral case. For the general case, we show, for instance, that globally solving either problem (P_X) or problem $(P_{Y^{\leqq}})$ essentially also globally solves the other problem, and that, for any feasible point $\overline{y}$ for problem $(P_{Y^{\leqq}})$, either $g(y) < g(\overline{y})$ for some $y \in Y^{\leqq}$ or $\overline{y}$ satisfies a condition that is necessary, but not sufficient, for it to be a local optimal solution for problem $(P_{Y^{\leqq}})$. For the convex and polyhedral cases, we show stronger

results. For example, we show for the convex case that any global optimal solution for problem $(P_{Y^{\leqq}})$ must lie on the boundary of $Y^{\leqq}$, that the objective function g in problem $(P_{Y^{\leqq}})$ is strictly pseudoconcave on $Y^{\leqq}$, and, when $Y^{\leqq}$ is closed and contains at least one extreme point, that problem $(P_{Y^{\leqq}})$ has an extreme point global optimal solution.

The analysis of the general case of problem $(P_{Y^{\leqq}})$ is given in Section 4.2. Section 4.3 provides analytical results for both the convex and polyhedral cases of problem $(P_{Y^{\leqq}})$.

4.2. Results for the General Case of Problem $(P_{Y^{\leqq}})$

Notice under the assumptions made in Section 4.1 for problem (P_X) that $Y^{\leqq}$ is a nonempty subset of $\boldsymbol{R}_+^p := \left\{z \in \boldsymbol{R}^p \middle| z > 0\right\}$. When $Y^{\leqq}$ satisfies this condition, we obtain what we will call the general case of problem $(P_{Y^{\leqq}})$.

It is important to establish that by solving the general case outcome-space formulation $(P_{Y^{\leqq}})$ of problem (P_X), a global optimal solution for problem (P_X) can be recovered. The following result, by showing that problems (P_X) and $(P_{Y^{\leqq}})$ are equivalent in a certain sense, immediately establishes this fact.

Theorem 4.2.1. (a) If x^* is a global optimal solution for problem (P_X), then $y^* = f(x^*)$ is a global optimal solution for problem $(P_{Y^{\leqq}})$. Furthermore, $v_y = v_x$.

(b) Problem $(P_{Y^{\leqq}})$ has at least one global optimal solution. Furthermore, if y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$, then any $x^* \in X$ such that $f(x^*) \leqq y^*$ is a global optimal solution for problem (P_X).

Proof. (a) Let x^* be a global optimal solution for problem (P_X), and set $y^* = f(x^*)$. From (4.1) and (4.2), this implies that $y^* \in Y^{\leqq}$ and that

$$g(y^*) = \prod_{j=1}^{p} f_j(x^*) = v_x.$$

Therefore, $v_y \leqq v_x$. If $g(\bar{y}) < v_x$ were to hold for some $\bar{y} \in Y^{\leqq}$, then, from (4.1) and (4.2), there would exist an $\bar{x} \in X$ such that

$$0 < \prod_{j=1}^{p} f_j(\bar{x}) \leqq g(\bar{y}) < v_x,$$

which contradicts the definition of v_x. Therefore, $g(y) \geqq v_x$ for all $y \in Y^{\leqq}$. This implies that $v_y \geqq v_x$. Since $v_y \leqq v_x = g(y^*)$, and $y^* \in Y^{\leqq}$, it follows that $v_y = v_x = g(y^*)$, and y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$.

(b) By assumption, we may choose a global optimal solution for problem (P_X). From part (a), this implies that problem $(P_{Y^{\leqq}})$ has at least one global optimal solution. Suppose that y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$. Since $y^* \in Y^{\leqq}$, (4.1) implies that we may choose an arbitrary $x^* \in X$ such that $f(x^*) \leqq y^*$. Then, from (4.2), since $0 < f(x^*)$,

$$\prod_{j=1}^{p} f_j(x^*) \leqq g(y^*).$$

Since $x^* \in X$ and y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$, this implies that

$$v_x \leqq \prod_{j=1}^{p} f_j(x^*) \leqq v_y. \tag{4.3}$$

From part (a), $v_y = v_x$. By (4.3), this implies that $\prod_{j=1}^{p} f_j(x^*) = v_x$. Since $x^* \in X$, it follows that x^* is a global optimal solution for problem (P_X). □

Suppose in the general case of problem $(P_{Y^{\leqq}})$ that a point $\bar{y} \in Y^{\leqq}$ has been generated. For algorithmic purposes, it may be valuable to have a tool for finding an alternate point $\tilde{y} \in Y^{\leqq}$ that satisfies $g(\tilde{y}) < g(\bar{y})$, if such a point exists. The next result gives an idea for potentially helping to create such a tool. To prove this result, we need the following lemma. This lemma will also be useful in proving several other results later in this chapter.

Lemma 4.2.1. Assume that $\bar{y} \in Y^{\leqq}$. Then, for any $y \in Y^{\leqq}$,

$$(1/p)\langle \nabla g(\bar{y}), y \rangle = g(\bar{y})(1/p)\sum_{j=1}^{p}(y_j/\bar{y}_j),$$

and

$$g(\bar{y})(1/p)\sum_{j=1}^{p}(y_j/\bar{y}_j) \geqq g(\bar{y})[g(y)/g(\bar{y})]^{1/p},$$

with equality holding in the latter relationship iff, for some constant $M > 0$, $y_j = M\bar{y}_j$, $j = 1, 2, \ldots, p$.

Proof. Choose an arbitrary point $\bar{y} \in Y^{\leqq}$. Suppose that $y \in Y^{\leqq}$. Then, by (4.2), since $Y^{\leqq} \subseteq R_+^p$, $g(\bar{y}) > 0$. By definition of g,

$$(1/p)\langle \nabla g(\bar{y}), y \rangle = (1/p)\sum_{j=1}^{p}\left(\prod_{k \neq j} \bar{y}_k\right) y_j$$

$$= (1/p)\sum_{j=1}^{p}[g(\bar{y})/\bar{y}_j]y_j$$

$$= g(\bar{y})\sum_{j=1}^{p}(1/p)(y_j/\bar{y}_j). \tag{4.4}$$

Since $(1/p)\geqq 0$, $(y_j/\bar{y}_j)>0$ for each $j=1,2,\ldots,p$, and $p(1/p)=1$, the arithmetic-geometric mean inequality (Duffin, Peterson, and Zener 1967) implies that

$$\sum_{j=1}^{p}(1/p)(y_j/\bar{y}_j)\geqq[g(y)/g(\bar{y})]^{1/p},$$

with equality holding iff for some constant $M>0$, $y_j = M\bar{y}_j$ for each $j=1,2,\ldots,p$.

Together with (4.4), since $g(\bar{y})\geqq 0$, this implies the desired results. □

Theorem 4.2.2. Assume that $\bar{y}\in Y^{\leqq}$. If

$$1.0 > \inf_{y\in Y^{\leqq}}(1/p)\sum_{j=1}^{p}(y_j/\bar{y}_j), \tag{4.5}$$

then $g(y)<g(\bar{y})$ for some $y\in Y^{\leqq}$. In particular, if $\tilde{y}$ achieves the infimum in (4.5), then $g(\tilde{y})<g(\bar{y})$.

Proof. Suppose that $\bar{y}\in Y^{\leqq}$. If (4.5) holds, then for some $\tilde{y}\in Y^{\leqq}$,

$$1.0 > (1/p)\sum_{j=1}^{p}(\tilde{y}_j/\bar{y}_j). \tag{4.6}$$

Since $g(\bar{y})>0$, this implies that

$$g(\bar{y}) > g(\bar{y})(1/p)\sum_{j=1}^{p}(\tilde{y}_j/\bar{y}_j). \tag{4.7}$$

From Lemma 4.2.1, since $\tilde{y}\in Y^{\leqq}$, we know that

$$g(\bar{y})(1/p)\sum_{j=1}^{p}(\tilde{y}_j/\bar{y}_j)\geqq g(\bar{y})[g(\tilde{y})/g(\bar{y})]^{1/p}. \tag{4.8}$$

Since $g(\bar{y})>0$, together (4.7) and (4.8) imply that

$$1.0 > [g(\tilde{y})/g(\bar{y})]^{1/p}.$$

Because $g(\bar{y}) > 0$, this implies that $g(\tilde{y}) < g(\bar{y})$. Therefore, $g(y) < g(\bar{y})$ for some $y \in Y^{\leq}$. Since, for any point $\tilde{y}$ that achieves the infimum in (4.5), (4.6) is also satisfied, the argument above also implies that if $\tilde{y}$ achieves the infimum in (4.5), then $g(\tilde{y}) < g(\bar{y})$. □

Notice that when $\bar{y} \in Y^{\leq}$, the infimum in (4.5) is either less than 1.0 or equal to 1.0. From Theorem 4.2.2, when this infimum is less than 1.0, a point y in $Y^{\leq}$ such that $g(y) < g(\bar{y})$ exists. In particular, in this case $\bar{y}$ is not a global optimal solution for problem $(P_{Y^{\leq}})$. The next result covers the case when the infimum in (4.5) equals 1.0.

Theorem 4.2.3. Assume that $\bar{y} \in Y^{\leq}$. If

$$1.0 = \inf_{y \in Y^{\leq}} (1/p) \sum_{j=1}^{p} (y_j / \bar{y}_j), \tag{4.9}$$

then $\bar{y}$ is an optimal solution to

$$v_d = \min_{y \in Y^{\leq}} \langle \nabla g(\bar{y}), y - \bar{y} \rangle, \tag{4.10}$$

and $v_d = 0$.

Proof. From (4.9), since $\bar{y} \in Y^{\leq}$, the infimum in (4.9) is achieved at $y = \bar{y}$. By Lemma 4.2.1, since $g(\bar{y})$ is a positive constant, this implies that $\bar{y}$ also minimizes $(1/p)\langle \nabla g(\bar{y}), y \rangle$ over $Y^{\leq}$. Since $(1/p)$ is a positive constant and $-\langle \nabla g(\bar{y}), \bar{y} \rangle$ is a constant, it is easy to see that this implies that $\bar{y}$ is an optimal solution to (4.10) and $v_d = 0$. □

A point $\bar{y} \in Y^{\leqq}$ is a local optimal solution for problem $(P_{Y^{\leqq}})$ when there exists an $\varepsilon > 0$ such that for each $y \in Y^{\leqq}$ for which $\|y - \bar{y}\| \leqq \varepsilon$, $g(y) \geqq g(\bar{y})$. From Theorem 4.2.3, when $\bar{y} \in Y^{\leqq}$ and (4.9) holds, then, for any $y \in Y^{\leqq}$, if there is a $\delta > 0$ such that $d := (y - \bar{y})$ satisfies $\bar{y} + \lambda d \in Y^{\leqq}$ for all λ such that $0 < \lambda \leqq \delta$, the directional derivative of g at $\bar{y}$ in the direction d will be nonnegative, i.e., $\langle \nabla g(\bar{y}), d \rangle \geqq 0$. From Bazaraa, Sherali and Shetty (1993), this is a necessary, but not sufficient condition for $\bar{y}$ to be a local (or global) optimal solution for problem $(P_{Y^{\leqq}})$.

4.3. Results for Convex and Polyhedral Cases of Problem $(P_{Y^{\leqq}})$

When $Y^{\leqq}$, in addition to being a nonempty subset of $\boldsymbol{R}_+^p$, is a convex set, then we obtain what we will call the convex case of problem $(P_{Y^{\leqq}})$. Similarly, when $Y^{\leqq}$, in addition to being a nonempty subset of $\boldsymbol{R}_+^p$, is a polyhedron, then we obtain what we will call the polyhedral case of problem $(P_{Y^{\leqq}})$. Each of these types of outcome-space versions of problem (P_X) arises from a broad class of decision space problems, as shown by the next result.

Theorem 4.3.1. When X is a convex set and, for each $j = 1, 2, \ldots, p$, f_j is a convex function on X, we obtain the convex case of problem $(P_{Y^{\leqq}})$. When X is a polyhedron and, for each $j = 1, 2, \ldots, p$, f_j is linear on $\boldsymbol{R}^n$, we obtain the polyhedral case of problem $(P_{Y^{\leqq}})$.

Proof. Assume, in addition to the assumptions made in Section 4.1 on X and on f_j, $j = 1, 2, \ldots, p$, that X is a convex set and that, for each $j = 1, 2, \ldots, p$, f_j is a

convex function on X. We will show that $Y^{\leqq}$ is a convex set. Choose any $y^1, y^2 \in Y^{\leqq}$. From (4.1), since $y^1, y^2 \in Y^{\leqq}$, we may choose $x^1, x^2 \in X$ such that $f_j(x^1) \leqq y^1$ and $f_j(x^2) \leqq y^2$, $j = 1, 2, \ldots, p$. Suppose that $\lambda \in R$ and $0 \leqq \lambda \leqq 1$. Then, since $\lambda \geqq 0$ and $(1-\lambda) \geqq 0$, for each $j = 1, 2, \ldots, p$,

$$\lambda f_j(x^1) + (1-\lambda) f_j(x^2) \leqq \lambda y^1 + (1-\lambda) y^2. \tag{4.11}$$

By the convexity of f_j, $j = 1, 2, \ldots, p$, on the convex set X, if we set $\bar{x} = \lambda x^1 + (1-\lambda) x^2$, then

$$\bar{x} \in X, \tag{4.12}$$

and, for each $j = 1, 2, \ldots, p$,

$$f_j(\bar{x}) \leqq \lambda f_j(x^1) + (1-\lambda) f_j(x^2). \tag{4.13}$$

From (4.11)-(4.13), $f(\bar{x}) \leqq \lambda y^1 + (1-\lambda) y^2$, where $\bar{x} \in X$. Since $y^1, y^2 \in Y^{\leqq}$, $y^i \leqq \hat{y}$ holds for each $i = 1, 2$. As a result, since $\lambda, (1-\lambda) \geqq 0$, $\lambda y^1 + (1-\lambda) y^2 \leqq \hat{y}$. The conditions for $\lambda y^1 + (1-\lambda) y^2$ to belong to $Y^{\leqq}$ are thus satisfied. By the choices of y^1, y^2 and λ, this implies that Y is a convex set.

Now suppose, in addition to the assumptions made in Section 4.1 on X and on f_j, $j = 1, 2, \ldots, p$, that X is a polyhedron and that, for each $j = 1, 2, \ldots, p$, f_j is a linear function on R^n. We will show that $Y^{\leqq}$ is a polyhedron. By definition, since X is a polyhedron, there exists a finite number q of linear functions g_j, $j = 1, 2, \ldots, q$, on R^n, and real numbers b_j, $j = 1, 2, \ldots, q$, such that

$$X = \{x \in R^n \mid g_j(x) \leqq b_j, j = 1, 2, \ldots, q\}.$$

Let $Z \subseteq \boldsymbol{R}^{n+p}$ be defined as the set of all solutions (x, y) to the system of linear inequalities (4.14)-(4.16) given by

$$f_j(x) - y_j \leqq 0, \qquad j = 1, 2, \ldots, p, \tag{4.14}$$

$$y_j \leqq \hat{y}_j, \qquad j = 1, 2, \ldots, p, \tag{4.15}$$

$$g_j(x) \leqq b_j, \qquad j = 1, 2, \ldots, q. \tag{4.16}$$

Then, by definition, Z is a polyhedron in $\boldsymbol{R}^{n+p}$. Let A be the $p \times (n+p)$ matrix whose first n columns each equal $0 \in \boldsymbol{R}^p$ and whose last p columns together form the $p \times p$ identity matrix. Then, from (4.1) and the definition of Z, $Y = AZ$. From Rockafellar (1970, Theorem 19.3), $Y^{\leqq}$ is a polyhedron in $\boldsymbol{R}^p$. □

In convex cases of problem $(P_{Y^{\leqq}})$ (and thus, in polyhedral cases as well), certain locations within $Y^{\leqq}$ for seeking global optimal solutions can be specified. For instance, we have the following result.

Theorem 4.3.2. Suppose that problem $(P_{Y^{\leqq}})$ satisfies the conditions for the convex case. Then:

(a) Any global optimal solution for problem $(P_{Y^{\leqq}})$ belongs to the boundary of $Y^{\leqq}$.

(b) If $Y^{\leqq}$ is closed and contains at least one extreme point, then there exists at least one global optimal solution for problem $(P_{Y^{\leqq}})$ that is an extreme point of $Y^{\leqq}$.

Proof. Assume that $Y^{\leqq}$, in addition to being a nonempty subset of $\boldsymbol{R}_+^p$, is a convex set, i.e., that we have the convex case for problem $(P_{Y^{\leqq}})$. Then, from Theorem 4.2.1, problem $(P_{Y^{\leqq}})$ has at least one global optimal solution.

(a) To show this part of the theorem, let y^* denote an arbitrary global optimal solution to problem $(P_{Y^{\leqq}})$. Suppose that y^* is not on the boundary of $Y^{\leqq}$. By the choice of y^* and since $Y^{\leqq}$ is a convex set, $Y^{\leqq}$ has a nonempty interior. Therefore, y^* must belong to the interior of $Y^{\leqq}$. From (4.1), this implies that for some $\bar{x} \in X$, $f(\bar{x}) < y^*$ must hold. By assumption, since $\bar{x} \in X$, $f(\bar{x}) > 0$. Therefore, if we set $\bar{y} = f(\bar{x})$, it follows that $\bar{y} \in Y^{\leqq}$ and

$$\prod_{j=1}^{p} \bar{y}_j < \prod_{j=1}^{p} y_j^*.$$

From (4.2), this contradicts the global optimality of y^* in problem $(P_{Y^{\leqq}})$. Therefore, y^* must belong to the boundary of $Y^{\leqq}$.

(b) From the discussion in Section 3.2, since $Y^{\leqq}$ is a nonempty convex set and, for each $j = 1, 2, \ldots, p$, the function $h_j(y) = y_j$ is positive and concave on $Y^{\leqq}$, the global optimal solution set for problem $(P_{Y^{\leqq}})$ is identical to the global optimal solution set for the problem

$$(\hat{P}_{Y^{\leqq}}) \quad \min \hat{g}(y), \text{ s.t. } y \in Y^{\leqq},$$

where, for each $y \in Y^{\leqq}$, $\hat{g} : Y^{\leqq} \to R$ is the concave function defined by

$$\hat{g}(y) = \left[\prod_{j=1}^{p} y_j \right]^{1/p}.$$

Since $Y^{\leqq}$ is a nonempty, closed convex set with at least one extreme point, from Rockafellar (1970, Corollary 18.5.3), it is easy to see that $Y^{\leqq}$ can contain no lines. Furthermore, since problem $(P_{Y^{\leqq}})$ has at least one global optimal solution, problem $(\hat{P}_{Y^{\leqq}})$

also has at least one global optimal solution. By Rockafellar (1970, Corollary 32.3.1), since $\hat{g}$ is a concave function on $Y^{\leqq}$, the latter two statements imply that problem $(\hat{P}_{Y^{\leqq}})$ has at least one global optimal solution $\bar{y}$ that is an extreme point of $Y^{\leqq}$. Because the optimal solution sets of problems $(P_{Y^{\leqq}})$ and $(\hat{P}_{Y^{\leqq}})$ coincide, this completes the proof. □

Suppose that $Y^{\leqq}$ is a nonempty, closed convex subset of $\boldsymbol{R}^p_+$, and that $Y^{\leqq}$ contains at least one extreme point. Then, from Theorem 4.3.2, there will exist at least one global optimal solution for problem $(P_{Y^{\leqq}})$ that is an extreme point of $Y^{\leqq}$, and all global optimal solutions for problem $(P_{Y^{\leqq}})$ will lie on the boundary of $Y^{\leqq}$. Neither of these properties, however, is necessarily shared by the decision set-based problem (P_X) whose outcome-space reformulation yields problem $(P_{Y^{\leqq}})$. The following example demonstrates this.

Example 4.3.1. Let $p=2$, $X=\left\{(x_1,x_2)^T \in \boldsymbol{R}^2 \middle| 0 \leqq x_i \leqq 6, i=1,2\right\}$,

$$f_1(x_1,x_2)=(x_1-1)^2+1,$$

and

$$f_2(x_1,x_2)=(x_2-2)^2+1$$

in problem (P_X). Then X is a nonempty, convex set and for each $i=1,2$, f_i is a convex, positively-valued function on X. Therefore, by Theorem 4.3.1, the problem $(P_{Y^{\leqq}})$ obtained by formulating the outcome-space version of problem (P_X) is guaranteed to satisfy the conditions of the convex case for problem $(P_{Y^{\leqq}})$. Furthermore, it is not difficult to show, in this case, that $Y^{\leqq}$ is compact. Thus, $Y^{\leqq}$ is closed and contains at

least one extreme point. It is easy to see that the unique global optimal solution to problem $(P_{Y^{\leqq}})$ is $(y^*)^T = (1,1)$ which, as guaranteed by Theorem 4.3.2, is an extreme point of $Y^{\leqq}$ (and is thus on the boundary of $Y^{\leqq}$). On the other hand, the only global optimal solution to problem (P_X) is $(x^*)^T = (1,2)$, yet x^* is neither on the boundary of X nor is it an extreme point of X.

To present the next result, we need to define two types of functions.

Definition 4.3.1. Let $Z \subseteq R^n$ be a nonempty convex set, and let $h: Z \to R$. The function h is said to be quasiconcave on Z when for each $z^1, z^2 \in Z$ and $\lambda \in R$ such that $0 < \lambda < 1$,

$$h[\lambda z^1 + (1-\lambda)z^2] \geq \min\{h(z^1), h(z^2)\}.$$

Definition 4.3.2. Let W be an open set in R^n that contains $Z \subseteq R^n$, and let $h: W \to R$. The function h is said to be strictly pseudoconcave over Z when h is differentiable over Z and, for each distinct $z^1, z^2 \in Z$, if $\langle \nabla h(z^1), z^2 - z^1 \rangle \leqq 0$, then $h(z^2) < h(z^1)$.

It is well known that a differentiable, quasiconcave function $h: Z \to R$ need not be strictly pseudoconcave over Z. For a discussion of quasiconcave and strictly pseudoconcave functions see, for example, Bazaraa, Sherali and Shetty (1993).

From Konno and Kuno (1995, p. 379), we know that when $Y^{\leqq}$ is a convex set, since $Y^{\leqq} \subseteq R_+^p$, $g: Y^{\leqq} \to R$ defined by (4.2) is quasiconcave on $Y^{\leqq}$. Thus, in the convex case, problem $(P_{Y^{\leqq}})$ is a minimization of a quasiconcave function over a convex set. In fact, however, we have the following even stronger result.

Theorem 4.3.3. Suppose that problem $(P_{Y^{\leqq}})$ satisfies the conditions for the convex case. Then, in this problem, g is a strictly pseudoconcave function over the convex set $Y^{\leqq}$.

Proof. The set $Y^{\leqq}$ is a convex set by definition of the convex case for problem $(P_{Y^{\leqq}})$. To show that g is strictly pseudoconcave over $Y^{\leqq}$, notice first that by (4.2), g can be considered to be well defined over the open set R_+^p. Also notice that g is differentiable over R_+^p and, thus, over $Y^{\leqq} \subseteq R_+^p$.

Suppose now that y^1 and y^2 are distinct points in $Y^{\leqq}$ that satisfy $\langle \nabla g(y^1), y^2 - y^1 \rangle \leqq 0$. Then, from (4.2), we obtain

$$0 \geqq \langle \nabla g(y^1), y^2 - y^1 \rangle = \sum_{k=1}^{p} \left(\prod_{j \neq k} y_j^1 \right) (y_k^2 - y_k^1)$$

$$= \sum_{k=1}^{p} \left[\prod_{j=1}^{p} y_j^1 / y_k^1 \right] y_k^2 - \sum_{k=1}^{p} g(y^1)$$

$$= \sum_{k=1}^{p} [g(y^1)](y_k^2 / y_k^1) - pg(y^1)$$

$$= g(y^1) \left[\sum_{k=1}^{p} (y_k^2 / y_k^1) - p \right]. \tag{4.17}$$

By multiplying both sides of (4.17) by $(1/p)$ and rearranging, we obtain that

$$g(y^1) \geqq g(y^1)(1/p) \sum_{k=1}^{p} (y_k^2 / y_k^1). \tag{4.18}$$

From Lemma 4.2.1,

$$g(y^1)(1/p) \sum_{k=1}^{p} (y_k^2 / y_k^1) \geqq g(y^1) [g(y^2) / g(y^1)]^{1/p}, \tag{4.19}$$

with equality holding iff, for some $M > 0$, $y_k^2 = M\, y_k^1$, $k = 1, 2, \ldots, p$. There are two cases to consider.

Case (i): There is no $M > 0$ such that $y_k^2 = M\, y_k^1$, $k = 1, 2, \ldots, p$. Then, in (4.19), strict inequality holds, so that from (4.18) and (4.19),

$$g(y^1) > g(y^1)\left[g(y^2)/g(y^1)\right]^{1/p}.$$

Since $g(y^1) > 0$, this implies that $g(y^2) < g(y^1)$.

Case (ii): For some $M > 0$, $y_k^2 = M\, y_k^1$, $k = 1, 2, \ldots, p$. If we choose such an M, then (4.19) holds as an equality. Thus, from (4.19) and the choice of M, we obtain that

$$g(y^1)(1/p)\sum_{k=1}^{p}(y_k^2/y_k^1) = g(y^1)\left[g(y^2)/g(y^1)\right]^{1/p} \tag{4.20}$$

and that

$$g(y^2) = M^p\, g(y^1), \tag{4.21}$$

respectively. Since $g(y^1) > 0$, together (4.18), (4.20) and (4.21) imply that

$$g(y^1) \geq g(y^1)(M^p)^{1/p} = M\, g(y^1).$$

Dividing through by $g(y^1) > 0$ yields $M \leq 1$. Notice that $M \neq 1$, since, by assumption, y^1 and y^2 are distinct. Therefore $M < 1$. By (4.21), since $g(y^1), g(y^2) > 0$, this implies that $g(y^2) < g(y^1)$, and the proof is complete. □

Remark 4.3.1. Theorem 4.3.3 justifies and strengthens the claim of Sniedovich and Findlay (1995, p. 317) that when $Y^{\leq}$ is a convex subset of $\boldsymbol{R}_+^p$, $g : Y^{\leq} \to \boldsymbol{R}$ defined by (4.2) is differentiable and pseudoconcave on $Y^{\leq}$.

From Theorem 4.3.3, in the convex case, problem $(P_{Y^{\leqq}})$ is a global optimization problem involving the minimization of a strictly pseudoconcave function over a convex set $Y^{\leqq}$. Therefore, as in the general case, multiple local optimal solutions for problem $(P_{Y^{\leqq}})$ will generally exist that are not globally optimal.

From Theorem 3.2.1, we know that when $Y^{\leqq}$ is a nonempty, convex subset of R_+^p, the function $\hat{g}: Y^{\leqq} \to R$ defined, as in the proof of Theorem 4.3.2, by

$$\hat{g}(y) = [g(y)]^{1/p} \tag{4.22}$$

is concave where $g: Y^{\leqq} \to R$ is given by (2). By the next result, when the domain of $\hat{g}$ is restricted to an appropriate subset of $Y^{\leqq}$, a stronger statement can be made.

Theorem 4.3.4. Assume that $Y^{\leqq}$ is a nonempty, compact, convex subset of R_+^p. For any $a \in R^p$ and $b \in R$ such that $a > 0$ and $b > 0$, let $Z(a,b) = Y^{\leqq} \bigcap \{y \in R^p \mid \langle a, y \rangle = b\}$. Then $\hat{g}: Z(a,b) \to R$ defined for each $y \in Z(a,b)$ by (4.22) is a strictly concave function for any $a \in R^p$ and $b \in R$ such that $a > 0$ and $b > 0$.

Proof. Assume that $y^1, y^2 \in Z(a,b)$ and $y^1 \neq y^2$, where $a \in R^p$, $b \in R$, $a > 0$, and $b > 0$. Since $Z(a,b)$ is an intersection of two convex sets, it is itself a convex set. Therefore, if we choose $\lambda \in R$ such that $0 < \lambda < 1$, then

$$z := \lambda y^1 + (1-\lambda) y^2 \in Z(a,b).$$

Also, by (2) and (4.22),

$$\hat{g}(z) = \left[\prod_{j=1}^{p} [\lambda y_j^1 + (1-\lambda) y_j^2]\right]^{1/p}. \tag{4.23}$$

From Polya and Szego (1972),

$$\left[\prod_{j=1}^{p}\left[\lambda y_j^1+(1-\lambda)y_j^2\right]\right]^{1/p} \geqq \left(\lambda^p \prod_{j=1}^{p} y_j^1\right)^{1/p} + \left[(1-\lambda)^p \prod_{j=1}^{p} y_j^2\right]^{1/p}, \tag{4.24}$$

with equality holding iff $\lambda y_j^1 = K(1-\lambda)y_j^2$, $j=1,2,\ldots,p$, for some positive constant K.

Since

$$\left(\lambda^p \prod_{j=1}^{p} y_j^1\right)^{1/p} = \lambda \hat{g}(y^1)$$

and

$$\left[(1-\lambda)^p \prod_{j=1}^{p} y_j^2\right]^{1/p} = (1-\lambda)\hat{g}(y^2),$$

(4.23) and (4.24) will imply the desired result if we can show that no $K>0$ exists such that

$$\lambda y_j^1 = K(1-\lambda)y_j^2, \quad j=1,2,\ldots,p. \tag{4.25}$$

Notice that since $y^1 \neq y^2$, $K_\lambda := [\lambda/(1-\lambda)]$ does not satisfy (4.25).

Suppose, to the contrary, that for some $K>0$, (4.25) is satisfied. Then from (4.25) it follows that

$$y^1 = K[(1-\lambda)/\lambda]y^2. \tag{4.26}$$

Since $y^1, y^2 \in Z(a,b)$,

$$\langle a, y^1\rangle = \langle a, y^2\rangle = b. \tag{4.27}$$

Substituting for y^1 in (4.27) via (4.26), we obtain

$$K[(1-\lambda)/\lambda]\langle a, y^2\rangle = \langle a, y^2\rangle = b.$$

Solving here for K, we obtain that $K = [\lambda/(1-\lambda)]$. Since $K = K_\lambda = [\lambda/(1-\lambda)]$ does not satisfy (4.25), this contradiction concludes the proof. □

It is important to notice that the counterpart of Theorem 4.3.4 in the decision space does not hold, even in the polyhedral case. In particular, suppose that $X \subseteq \boldsymbol{R}^n$ is a nonempty, compact polyhedron and, for each $j = 1, 2, \ldots, p$, that there exists a $c_j \in \boldsymbol{R}^n$ such that $f_j(x) = \langle c_j, x \rangle > 0$ for all $x \in X$. Then, although the function $h : X \to \boldsymbol{R}$ defined for each $x \in X$ by

$$h(x) = \left[\prod_{j=1}^{p} \langle c_j, x \rangle \right]^{1/p} \tag{4.28}$$

is concave (see Theorem 3.2.1), the function $h : X(a,b) \to \boldsymbol{R}$ need not be strictly concave, where $a \in \boldsymbol{R}^p$, $b \in \boldsymbol{R}$, $a > 0$, $b > 0$, and

$$X(a,b) = \left\{ x \in X \middle| \sum_{j=1}^{p} a_j \langle c_j, x \rangle = b \right\}.$$

The following example illustrates this observation.

Example 4.3.2. Let

$$X = \left\{ (x_1, x_2)^T \in \boldsymbol{R}^2 \middle| 0.5 \leqq x_i \leqq 4.0, i = 1, 2 \right\},$$

and let $f_j(x_1, x_2) = \langle (1,1), (x_1, x_2) \rangle$, $j = 1, 2$. Then X is a nonempty, compact polyhedron and, for each $j = 1, 2$, f_j is positive and linear on X. As guaranteed by Theorem 3.2.1, $h : X \to \boldsymbol{R}$, which, by (4.28), is given by

$$h(x_1, x_2) = (x_1 + x_2),$$

is concave. However, if, for example, $a_1 = a_2 = 1$ and $b = 4$, then h is not strictly concave on

$$X(a,b) = \left\{(x_1, x_2)^T \middle| 0.5 \leqq x_i \leqq 4.0, i = 1,2, x_1 + x_2 = 4\right\}.$$

Consider now problem $(P_{Y^{\leqq}})$ when the conditions of the polyhedral case hold. Assume also that $Y^{\leqq}$ is a compact set, and that $\bar{y} \in Y^{\leqq}$. For algorithmic purposes, it may be quite useful in this case to develop tools for finding local optimal solutions for problem $(P_{Y^{\leqq}})$. These tools could then potentially be used to construct global solution algorithms for the problem that repeatedly move from a local optimal solution to an improved local optimal solution until a global optimal solution is found. The remaining results in this section are motivated, in part, by the desire to find such tools.

Notice that in the polyhedral case, the optimization problem in (9) is a linear program given by

$$(LP) \qquad \min (1/p)\sum_{j=1}^{p}\left(y_j/\bar{y}_j\right), \text{s.t. } y \in Y^{\leqq}.$$

Problem (*LP*) will have an optimal solution y^* that can be found, for instance, by the simplex method. Since $\bar{y} \in Y^{\leqq}$, the minimum value vmin in problem (*LP*) satisfies vmin $\leqq 1.0$. As a result, there are three possible cases for problem (*LP*). First, vmin < 1.0 may hold. Second, vmin = 1.0 may hold, with $\bar{y}$ being the unique optimal solution to problem (*LP*). Third, vmin = 1.0 may hold, with problem (*LP*) having multiple optimal solutions. In the first case, from Theorem 4.2.2, it follows that $g(y^*) < g(\bar{y})$, where y^* is any optimal solution to problem (*LP*), so that a more attractive feasible solution y^* to

problem $(P_{Y^{\leq}})$ than $\bar{y}$ has been found. To analyze the second case, we need the following two definitions and lemma.

Definition 4.3.3. A point $\bar{y} \in Y^{\leq}$ is a strict local optimal solution for problem $(P_{Y^{\leq}})$ when there exists an $\varepsilon > 0$ such that for each $y \in Y^{\leq}$ for which $y \neq \bar{y}$ and $\|y - \bar{y}\| < \varepsilon$, $g(y) > g(\bar{y})$.

Definition 4.3.4. Let Z be a nonempty convex set in R^n, and let $h: Z \to R$. The function h is said to be strongly quasiconcave on Z when for each $z^1, z^2 \in Z$ with $z^1 \neq z^2$, we have

$$h[\lambda z^1 + (1-\lambda)z^2] > \min\{h(z^1), h(z^2)\}$$

for each λ such that $0 < \lambda < 1$.

Lemma 4.3.1. Let Z be a nonempty convex set in R^n, and let $h: Z \to R$ be strongly quasiconcave. Suppose that z^i, $i = 1, 2, \ldots, k$, are distinct points in Z and that s is an element of the convex hull of z^i, $i = 1, 2, \ldots, k$, such that, for each $i = 1, 2, \ldots, k$, $s \neq z^i$. Then

$$h(s) > \min\{h(z^i) \mid i = 1, 2, \ldots, k\}.$$

Proof. The lemma is easy to prove using Definition 4.3.4 and induction. □

The following result analyzes the case where vmin = 1.0 and $\bar{y}$ is the unique optimal solution to problem (*LP*).

Theorem 4.3.5. Assume that problem $(P_{Y^{\leq}})$ satisfies the conditions for the polyhedral case, and that $Y^{\leq}$ is a compact set. Assume also that $\bar{y} \in Y^{\leq}$. Suppose that vmin = 1.0

and that $y = \bar{y}$ is the unique optimal solution to problem (*LP*). Then $\bar{y}$ is a strict local optimal solution for problem $(P_{Y^{\leq}})$.

Proof. Since $g(\bar{y}) > 0$ and $y = \bar{y}$ is the unique optimal solution to the problem

$$\min_{y \in Y^{\leq}} (1/p) \sum_{j=1}^{p} (y_j / \bar{y}_j),$$

$y = \bar{y}$ must also be the unique optimal solution to the problem

$$\min_{y \in Y^{\leq}} g(\bar{y})(1/p) \sum_{j=1}^{p} (y_j / \bar{y}_j).$$

Therefore, by Lemma 4.2.1, $y = \bar{y}$ is the unique optimal solution to the problem

$$\min_{y \in Y^{\leq}} (1/p) \langle \nabla g(\bar{y}), y \rangle.$$

Since $(1/p) > 0$ and $-\langle \nabla g(\bar{y}), \bar{y} \rangle$ is a constant, this implies that $y = \bar{y}$ is the unique optimal solution to the problem

$$\min_{y \in Y^{\leq}} \langle \nabla g(\bar{y}), y - \bar{y} \rangle. \tag{4.29}$$

Therefore, the optimal value of problem (4.29) is 0, and for all $y \in Y^{\leq}$ such that $y \neq \bar{y}$,

$$\langle \nabla g(\bar{y}), y - \bar{y} \rangle > 0. \tag{4.30}$$

Let $d^1, d^2, \ldots, d^k$ represent the directions of the edges of $Y^{\leq}$ emanating from the extreme point $\bar{y}$ of $Y^{\leq}$. From (4.30),

$$\langle \nabla g(\bar{y}), d^i \rangle > 0$$

for all $i = 1, 2, \ldots, k$. By Theorem 4.1.2 in Bazaraa, Sherali, and Shetty (1993), this implies that there exist positive reals δ_i, $i = 1, 2, \ldots, k$, such that

$$g(\bar{y} + \lambda_i d^i) > g(\bar{y}) \tag{4.31}$$

for each $\lambda_i \in (0, \delta_i)$. Let $\delta = 1/2 \min\{\delta_i | i = 1, 2, \ldots, k\}$, and consider the points $\bar{y}$ and

$$\bar{y} + \delta d^1, \ \bar{y} + \delta d^2, \ \ldots, \ \bar{y} + \delta d^k.$$

Then by definition of δ and (4.31),

$$g(\bar{y} + \delta d^i) > g(\bar{y}) \tag{4.32}$$

for each $i = 1, 2, \ldots, k$. Let z be any element of the convex hull of $\bar{y}$, $\bar{y} + \delta d^i$, $i = 1, 2, \ldots, k$, such that $z \neq \bar{y}$ and, for each $i = 1, 2, \ldots, k$, $z \neq \bar{y} + \delta d^i$. Since g is a strictly pseudoconcave function on $Y^{\leqq}$, it is also a strongly quasiconcave function on $Y^{\leqq}$ (Bazaraa, Sherali, Shetty 1993). As a result, by Lemma 4.3.1,

$$g(z) > \min\{g(\bar{y}), g(\bar{y} + \delta d^i), i = 1, 2, \ldots, k\}. \tag{4.33}$$

From (4.32) and (4.33), $g(z) > g(\bar{y})$. Since $\delta > 0$, this implies that there exists an $\varepsilon > 0$ sufficiently small so that if $z \in Y^{\leqq}$, $\|z - \bar{y}\| < \varepsilon$, and $z \neq \bar{y}$, then $g(z) > g(\bar{y})$. □

Under the assumptions of Theorem 4.3.5, if vmin = 1.0 but $y = \bar{y}$ is one of two or more optimal solutions to problem (*LP*), then $\bar{y}$ need <u>not</u> be a strict local optimal solution for problem ($P_{Y^{\leqq}}$). The following example illustrates this point.

<u>Example 4.3.3.</u> Let

$$Y^{\leqq} = \{(y_1, y_2)^T \in \boldsymbol{R}^2 | y_1 + y_2 \geqq 8, \ y_1 \leqq 7, \ y_2 \leqq 4\},$$

and let $\bar{y}^T = (4, 4)^T$. Then $Y^{\leqq}$ is a nonempty, compact polyhedron in $\boldsymbol{R}^2_+$, and the assumptions of Theorem 4.3.5 are satisfied. In this case, $\bar{y} \in Y^{\leqq}$ and $\bar{y}$ is an optimal solution to problem (*LP*). However, since $(y^\delta)^T = (4 + \delta, 4 - \delta)^T \in Y^{\leqq}$ and $g(y^\delta) < g(\bar{y})$ for all values of δ such that $0 < \delta \leq 3$, by Definition 4.3.3, $\bar{y}$ is not a strict local optimal

solution for problem $(P_{Y^{\leqq}})$. (In fact, $\bar{y}$ is not even a local optimal solution for problem $(P_{Y^{\leqq}})$.) Notice that problem (*LP*) in this case has multiple optimal solutions.

In the third case of problem (*LP*), vmin = 1.0 and problem (*LP*) has multiple optimal solutions. In this case, by the next result, as in the first case, an improved feasible solution for problem $(P_{Y^{\leqq}})$ is at hand. The proof of this result relies crucially on Theorem 4.3.4.

Theorem 4.3.6. Assume that problem $(P_{Y^{\leqq}})$ satisfies the conditions for the polyhedral case, and that $Y^{\leqq}$ is compact. Suppose that $\bar{y}$ is an optimal solution for problem (*LP*), and suppose that problem (*LP*) has multiple optimal solutions. Then, for any $y^* \neq \bar{y}$ that is an optimal solution for problem (*LP*), $g(y^*) < g(\bar{y})$ must hold.

Proof. Let $y^* \neq \bar{y}$ be an optimal solution to problem (*LP*). Then, since $g(\bar{y}) > 0$, y^* is also an optimal solution for the problem

$$\min g(\bar{y})(1/p)\sum_{j=1}^{p}(y_j/\bar{y}_j), \text{ s.t. } y \in Y^{\leqq}.$$

By Lemma 4.2.1, since $\bar{y} \in Y^{\leqq}$, this implies that y^* is an optimal solution to the problem

$$\min (1/p)\langle \nabla g(\bar{y}), y \rangle, \text{ s.t. } y \in Y^{\leqq}.$$

Since $-(1/p)\langle \nabla g(\bar{y}), \bar{y} \rangle$ is a fixed number, it follows that y^* is an optimal solution to the problem

$$\min (1/p)\langle \nabla g(\bar{y}), y - \bar{y} \rangle, \text{ s.t. } y \in Y^{\leqq}. \tag{4.34}$$

By assumption, $\bar{y}$ is an optimal solution for problem (*LP*). Therefore, the optimal value of problem (*LP*) equals 1.0. From Theorem 4.2.3, this implies that the optimal value of

problem (4.34) equals 0. As a result, since y^* is an optimal solution for problem (4.34), $\langle \nabla g(\bar{y}), y^* - \bar{y} \rangle = 0$. By the choice of y^*, it follows that

$$Y^*(LP) \subseteq \{ y \in Y \mid \langle \nabla g(\bar{y}), y - \bar{y} \rangle = 0 \}, \tag{4.35}$$

where $Y^*(LP)$ denotes the optimal solution set for problem (*LP*).

From Theorem 4.3.4, since $\nabla g(\bar{y}) > 0$ and $\langle \nabla g(\bar{y}), \bar{y} \rangle > 0$, the function $\hat{g}$ defined by (4.22) over the set $Y^* := \left(Y^{\leq} \cap \{ y \in \boldsymbol{R}^p \mid \langle \nabla g(\bar{y}), y \rangle = \langle \nabla g(\bar{y}), \bar{y} \rangle \} \right)$ is strictly concave. Notice, in addition, that $\hat{g}$ is differentiable on $\boldsymbol{R}^p_+$. By (4.35), $\bar{y}$ and y^* both belong to Y^*. From Bazaraa, Sherali, and Shetty (1993), since $y^* \neq \bar{y}$, the latter three sentences together imply that

$$\hat{g}(y^*) < \hat{g}(\bar{y}) + \langle \nabla \hat{g}(\bar{y}), y^* - \bar{y} \rangle.$$

From (2) and (4.22), this implies that

$$\begin{aligned}
[g(y^*)]^{1/p} &< [g(\bar{y})]^{1/p} + (1/p) \sum_{j=1}^{p} [g(\bar{y})^{1/p} / \bar{y}_j](y_j^* - \bar{y}_j) \\
&= [g(\bar{y})]^{1/p} + [g(\bar{y})^{1/p} / p] \sum_{j=1}^{p} [(y_j^* - \bar{y}_j) / \bar{y}_j] \\
&= [g(\bar{y})]^{1/p} + [g(\bar{y})]^{[(1-p)/p]} (1/p) \langle \nabla g(\bar{y}), y^* - \bar{y} \rangle \\
&= g(\bar{y})^{1/p},
\end{aligned}$$

where the last equation follows from the fact that $y^* \in Y^*$. As a result, $g(y^*) < g(\bar{y})$. □

Remark 4.3.2. Suppose that problem $(P_{Y^{\leq}})$ satisfies the conditions for the polyhedral case and that $Y^{\leq}$ is compact. Then, by using Theorem 4.3.6 and the discussion preceding it, a

finite guaranteed method can be described for finding a strict local optimal extreme point solution for problem $(P_{Y^{\leqq}})$. For instance, one such method is as follows.

Algorithm 4.3.1. Strict Local Optimal Extreme Point Solution Search

Initialization Step. Find an initial extreme point $\bar{y}$ of $Y^{\leqq}$.

Step 1. Find the optimal value vmin for linear program (*LP*). If vmin < 1.0, go to Step 2. If vmin = 1.0, go to Step 3.

Step 2. Find any optimal extreme point solution y^* for problem (*LP*). Set $\bar{y} = y^*$, and return to Step 1.

Step 3. If $\bar{y}$ is the unique optimal solution for problem (*LP*), then stop: The point $\bar{y}$ is a strict local optimal solution for problem $(P_{Y^{\leqq}})$. If problem (*LP*) has multiple optimal solutions, then continue.

Step 4. Find any optimal extreme point solution $y^* \neq \bar{y}$ for problem (*LP*). Set $\bar{y} = y^*$, and return to Step 1.

The method is finite because $Y^{\leqq}$ must have a finite number of extreme points. The methods of Benson (1998d) and Benson and Sun (1998) can be shown to be suitable for executing the initialization step and for solving the linear program (*LP*) in Steps 1, 2 and 4. Notice that the counterpart to this method for the decision set-based problem (P_X) is not guaranteed to succeed. This is shown by the following example.

Example 4.3.4. Let

$$X = \left\{(x_1, x_2)^T \in \boldsymbol{R}^2 \middle| 0.5 \leqq x_1,\ x_2 \leqq 3.0,\ x_1 + x_2 \geqq 2.0\right\},$$

and, for each $x \in \boldsymbol{R}^2$, let

$$f_1(x) = x_1 + x_2, \quad f_2(x) = 4x_1 + 4x_2.$$

Then X is a nonempty, compact polyhedron, and, for each $j = 1, 2$, $f_j(x)$ is a linear function with positive values for all $x \in X$. If we generate the initial extreme point $(x^1)^T = (1.5, 0.5)$ of X, for example, then the linear program

$$\min (1/2)\sum_{j=1}^{2} [f_j(x)/f_j(x^1)], \text{ s.t. } x \in X$$

has optimal value 1.0, and it has exactly two extreme point optimal solutions. These are x^1 and $(x^2)^T = (0.5, 1.5)^T$. Since $f_j(x^1) = f_j(x^2)$, $j = 1, 2$, the counterpart procedure to the above method would in this case cycle without termination between generating the extreme point x^1 and generating the extreme point x^2. Neither of these extreme points is a strict local optimal solution for problem (P_X), where the definition of a strict local optimal solution $\bar{x}$ for problem (P_X) is given by Definition 4.3.3, with $\bar{x}$, X, (P_X), x and $\prod_{j=1}^{p} f_j$ replacing $\bar{y}$, $Y^{\leqq}$, $(P_{Y^{\leqq}})$, y and g, respectively.

4.4. Discussion

The analysis of the outcome-space problem formulation $(P_{Y^{\leqq}})$ of the multiplicative programming problem (P_X) yields several results. The key ones are as follows.

First, since globally solving problem $(P_{Y^{\leqq}})$ also essentially globally solves problem (P_X), and since $Y^{\leqq}$ generally lies in a significantly smaller space than X, there could be great computational gain to be derived by constructing global optimization

algorithms for problem (P_X) that work directly on problem $(P_{Y^{\leqq}})$ instead of on problem (P_X).

Second, the potential to create global solution algorithms for problem $(P_{Y^{\leqq}})$ is quite high. For instance, in the convex case where $Y^{\leqq}$ is closed and has at least one extreme point, the analysis has shown that problem $(P_{Y^{\leqq}})$ possesses at least one global optimal solution that is an extreme point of $Y^{\leqq}$. This result could potentially allow researchers to create algorithms for solving problem $(P_{Y^{\leqq}})$ that concentrate on searching among the extreme points of $Y^{\leqq}$ in ways similar to those used in existing global optimization algorithms for other nonconvex programming problems (Horst and Tuy 1993).

Third, it appears potentially possible to construct global solution algorithms for problem $(P_{Y^{\leqq}})$ that are based, at least in part, upon local optimal solution searches. Indeed, the analysis shows, for example, the potential to create search mechanisms for finding strict local optimal solutions for problem $(P_{Y^{\leqq}})$ in the polyhedral case if $Y^{\leqq}$ is bounded. Combined with ideas from global optimization such as relief indicator functions or cutting planes (Horst and Tuy 1993), this suggests that global solution algorithms for problem $(P_{Y^{\leqq}})$ might be possible wherein successive local optimal solutions of smaller and smaller objective function values are found until a global optimal solution is found and the search terminates.

CHAPTER 5
AN OUTCOME-SPACE CUTTING-PLANE ALGORITHM FOR LINEAR MULTIPLICATIVE PROGRAMMING

5.1. Introduction

The linear multiplicative programming problem may be written

$$(P_X) \qquad \min f(x)=\prod_{j=1}^{p}\langle c_j,x\rangle, \quad \text{s.t. } x\in X,$$

where $X \subseteq \boldsymbol{R}^n$ is a nonempty, compact polyhedron, $p \geqq 2$ is an integer, and, for each $j=1,2,\ldots,p,\ c_j \in \boldsymbol{R}^n$ satisfies $\langle c_j,x\rangle > 0$ for all $x\in X$.

For each $j=1,2,\ldots,p,$ let $\hat{y}_j \in \boldsymbol{R}$ satisfy

$$\hat{y}_j > \max \langle c_j,x\rangle, \quad \text{s.t. } x\in X,$$

and let $\hat{y}\in \boldsymbol{R}^p$ denote the vector with jth entry equal to $\hat{y}_j,\ j=1,2,\ldots,p$. Let C denote the $p\times n$ matrix whose jth row equals $c_j^T,\ j=1,2,\ldots,p$. Recall from Chapter 1 that one of the more direct reformulations of problem (P_X) as an outcome-space problem is given by

$$(P_{Y^{\leqq}}) \qquad \min g(y)=\prod_{j=1}^{p} y_j, \quad \text{s.t. } y\in Y^{\leqq},$$

where

$$Y^{\leqq}=\left\{y\in \boldsymbol{R}^p \middle| Cx\leqq y\leqq \hat{y} \text{ for some } x\in X\right\}.$$

Closely related to this reformulation is the problem

$$(P_Y) \qquad \min g(y) = \prod_{j=1}^{p} y_j, \quad \text{s.t. } y \in Y,$$

where

$$Y = \left\{ y \in \boldsymbol{R}^p \,\middle|\, y = Cx \text{ for some } x \in X \right\}.$$

In this chapter, we develop an outcome-space, cutting-plane algorithm for globally solving the linear multiplicative programming problem (P_X). To accomplish this, we use the framework of a pure cutting plane, decision set-based concave minimization method of Horst and Tuy (1993, pp. 175-184). We show how to adapt this method to solving the outcome-space formulation $(P_{Y^{\leq}})$ of problem (P_X) for a global, extreme point optimal solution. Because p is almost always smaller than n, often by several orders of magnitude, we expect that potentially considerable computational savings could be obtained by using the new outcome-space, pure cutting plane algorithm instead of a decision set-based approach. As a further computational enhancement, we also show that for purposes of implementation, the mechanics of the outcome-space, cutting-plane algorithm can be applied to the smaller problem (P_Y) instead of to problem $(P_{Y^{\leq}})$.

The next section gives some theoretical prerequisites that will help to present and justify the steps of the new algorithm. The new outcome-space, pure cutting plane algorithm for globally solving problem (P_X) is presented and analyzed in Section 5.3. Section 5.4 shows that to further enhance computational efficiency, in practice the new algorithm can be applied to the outcome-space reformulation (P_Y) instead of to problem $(P_{Y^{\leq}})$. A sample problem is globally solved with the new algorithm in Section 5.5, and some concluding remarks are given in the last section.

5.2. Theoretical Prerequisites

The outcome-space algorithm uses extreme point search and cutting planes to find a global optimal extreme point solution to problem $(P_{Y^{\leqq}})$. From this solution, a global optimal extreme point solution to the original multiplicative programming problem (P_X) can be easily recovered, as we shall see. The approach of the new algorithm relies on several theoretical results. In this section, we review or develop the necessary results of this type.

The theoretical prerequisites concern problems (P_X) and $(P_{Y^{\leqq}})$ and their relationships to one another. Before presenting these prerequisites, we must give some preliminary definitions.

Definition 5.2.1. Let W be an open set in $\boldsymbol{R}^n$ that contains $Z \subseteq \boldsymbol{R}^n$, and let $h : W \to \boldsymbol{R}$. The function h is called strictly pseudoconcave over Z when h is differentiable over Z and, for each pair of distinct points $z^1, z^2 \in Z$, if $\left\langle \nabla h(z^1), z^2 - z^1 \right\rangle \leqq 0$, then $h(z^2) < h(z^1)$.

From Bazaraa, Sherali, and Shetty (1993), a strictly pseudoconcave function $h : Z \to \boldsymbol{R}$ defined over a convex set Z is both pseudoconcave and quasiconcave over Z. The converse to this statement, however, is not true. For details, see Bazaraa, Sherali, and Shetty (1993).

Definition 5.2.2. (Bazaraa, Sherali, and Shetty 1993). A point $\bar{y} \in Y^{\leqq}$ is called a strict local optimal solution for problem $(P_{Y^{\leqq}})$ when there exists an $\varepsilon > 0$ such that for each $y \in Y^{\leqq}$ for which $y \neq \bar{y}$ and $\|y - \bar{y}\| < \varepsilon$, $g(y) > g(\bar{y})$.

Each of the next five results either is taken directly from or follows easily from results in Chapters 3 and 4. For each result, the appropriate reference is given.

Proposition 5.2.1. (Theorem 4.3.1). The feasible region $Y^{\leqq}$ for problem $(P_{Y^{\leqq}})$ is a nonempty polyhedral subset of $\boldsymbol{R}_+^p := \left\{ y \in \boldsymbol{R}^p \middle| y > 0 \right\}$.

Proposition 5.2.2. (Proposition 3.2.2). Problem (P_X) possesses a global optimal solution that is an extreme point of X.

Theorem 5.2.1. (Theorem 4.3.2). Problem $(P_{Y^{\leqq}})$ possesses a global optimal solution that is an extreme point of $Y^{\leqq}$.

Theorem 5.2.2. (Theorem 4.2.1). If y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$, then any $x^* \in X$ such that $Cx^* \leqq y^*$ is a global optimal solution for problem (P_X). Furthermore, the global optimal values of problems (P_X) and $(P_{Y^{\leqq}})$ are equal.

Theorem 5.2.3. (Theorem 4.3.3). The objective function g of problem $(P_{Y^{\leqq}})$ is a strictly pseudoconcave function over $\boldsymbol{R}_+^p$.

Notice also in Proposition 5.2.1 that $Y^{\leqq}$ is full dimensional and bounded. Additionally, notice in Theorem 5.2.2 that given y^* as defined there, for any $x^* \in X$ such that $Cx^* \leqq y^*$, x^* not only is a global optimal solution for problem (P_X), but it also satisfies $Cx^* = y^*$. Together with Theorem 5.2.2 and an observation given in Benson and Sun (1998), this implies that if y^* is a global optimal solution for problem $(P_{Y^{\leqq}})$, then the linear program

$$\min \sum_{i=1}^{p} u_i$$

$$\text{s.t.} \quad Cx - u = y^*,$$

$$u \geqq 0, \quad x \in X,$$

will have at least one optimal solution, and for any optimal basic solution $(x^*,u^*)=$ $(x^*,0)\in R^{n+p}$ to this linear program, x^* is an extreme point global optimal solution for problem (P_X). Thus, given a global optimal solution for problem $(P_{Y^{\leqq}})$, a global optimal extreme point solution for problem (P_X) can be easily recovered by solving a single linear program for a basic optimal solution.

Based upon Theorem 5.2.1, the outcome-space, cutting-plane algorithm to be presented confines its search to extreme points of the nonempty, compact polyhedron $Y^{\leqq}$. In fact, the algorithm searches only within a certain subset of the set of extreme points of $Y^{\leqq}$. This is justified in part by the following result.

Theorem 5.2.4. Any point that is a global optimal solution for problem $(P_{Y^{\leqq}})$ must also be an extreme point of $Y^{\leqq}$ and a strict local optimal solution for problem $(P_{Y^{\leqq}})$.

Proof. Either (i) problem $(P_{Y^{\leqq}})$ has a unique global optimal solution or (ii) problem $(P_{Y^{\leqq}})$ has multiple global optimal solutions.

Case (i): Problem $(P_{Y^{\leqq}})$ has a unique global optimal solution. Let $\bar{y}$ represent this solution. Then, by Theorem 5.2.1, $\bar{y}$ is an extreme point of $Y^{\leqq}$. Since $\bar{y}$ is the unique global optimal solution for problem $(P_{Y^{\leqq}})$, $g(y)> g(\bar{y})$ for all $y\in Y^{\leqq}$ such that $y\neq\bar{y}$. Therefore, for all $\varepsilon>0$ and for each $y\in Y^{\leqq}$ such that $y\neq\bar{y}$ and $\|y-\bar{y}\|<\varepsilon$, $g(y)> g(\bar{y})$. By Definition 5.2.2, this completes the proof for this case.

Case (ii): Problem $(P_{Y^{\leqq}})$ has multiple global optimal solutions. Let $\bar{y}$ be any nonextreme point of $Y^{\leqq}$. Consider the linear programming problem

$$\min (1/p)\sum_{j=1}^{p}\left(y_j/\bar{y}_j\right), \text{ s.t. } y \in Y^{\leqq}. \tag{5.1}$$

Notice that since $\bar{y} \in Y^{\leqq}$, the optimal value v of (5.1) satisfies $v \leqq 1$. Therefore, either $v<1$ or $v=1$.

If $v<1$, then, from Theorem 4.2.2, there exists some $y \in Y^{\leqq}$ such that $g(y)< g(\bar{y})$. Therefore, in this case, $\bar{y}$ is not a global optimal solution for problem $(P_{Y^{\leqq}})$.

If $v=1$, then, since $\bar{y}$ is a nonextreme point of $Y^{\leqq}$, the linear program (5.1) has multiple optimal solutions, including $\bar{y}$. In this case, by Theorem 4.3.6, for any $y^* \neq \bar{y}$ that an optimal solution for (5.1), $g(y^*)< g(\bar{y})$ must hold. Therefore, in this case, $\bar{y}$ is not a global optimal solution for problem $(P_{Y^{\leqq}})$.

Since either $v<1$ or $v=1$, the arguments above imply that no nonextreme point of $Y^{\leqq}$ can be a global optimal solution for problem $(P_{Y^{\leqq}})$. Therefore, each of the global optimal solutions for problem $(P_{Y^{\leqq}})$ must be an extreme point of $Y^{\leqq}$.

Let $\hat{y}$ be an extreme point optimal solution for problem $(P_{Y^{\leqq}})$. Then, since the number of extreme points of $Y^{\leqq}$ is finite (see Murty (1983), for e.g.), there must exist an $\varepsilon>0$ such that if $\|y-\hat{y}\|<\varepsilon$, $y \neq \hat{y}$, and $y \in Y^{\leqq}$, then y is not an extreme point of $Y^{\leqq}$. Choose such an ε and let $y \neq \hat{y}$ satisfy $\|y-\hat{y}\|<\varepsilon$, $y \in Y^{\leqq}$. Then y is not an extreme point of $Y^{\leqq}$. Since each global optimal solution for problem $(P_{Y^{\leqq}})$ must be an extreme

point of $Y^{\leqq}$, this implies that $g(y) > g(\hat{y})$. As a result, by Definition 5.2.2, $\hat{y}$ is a strict local optimal solution for problem $(P_{Y^{\leqq}})$, and the proof is complete. □

Recall that a point z in a set $Z \subseteq R^n$ is called an isolated point of Z when it is not a limit point of Z. From Theorem 5.2.4, we obtain the following result as an immediate consequence.

Corollary 5.2.1. Let $(Y^{\leqq})^*$ denote the set of global optimal solutions for problem $(P_{Y^{\leqq}})$. Then every point in $(Y^{\leqq})^*$ is an isolated point of $(Y^{\leqq})^*$.

To conclude this section, we state the following result. The proof of this result is immediate by definition.

Proposition 5.2.3. Let $\bar{y}$ be an extreme point global optimal solution for problem $(P_{Y^{\leqq}})$. Then, for each extreme point $y \in Y^{\leqq}$ that is adjacent to $\bar{y}$, $g(y) \geqq g(\bar{y})$ must hold.

5.3. Outcome-Space, Cutting-Plane Algorithm

To adapt the approach of the decision set-based Horst-Tuy concave minimization algorithm (Horst and Tuy (1993), pp. 175-184) to solving the outcome-space formulation $(P_{Y^{\leqq}})$ of problem (P_X), three tasks need to be iteratively executed. The first is a certain strict local optimal solution search, the second involves the construction of a cutting plane, and the third is a termination test. Before giving a formal statement of the new outcome-space algorithm, we will describe how the algorithm executes each of these three tasks.

5.3.1. Strict Local Optimal Solution Search

Motivated by Proposition 5.2.3 and Theorems 5.2.1 and 5.2.4, the strict local optimal solution search seeks an extreme point $\bar{y}$ of $Y^{\leqq}$ that is a strict local optimal solution for problem $(P_{Y^{\leqq}})$ and satisfies $g(y) \geqq g(\bar{y})$ for each extreme point y of $Y^{\leqq}$ that is adjacent to $\bar{y}$. This search relies heavily upon repeatedly solving the linear program

$$\min (1/p)\sum_{j=1}^{p}(y_j/\bar{y}_j), \text{ s.t. } y \in Y^{\leqq} \tag{5.2}$$

as $\bar{y}$ is set equal to the values of various extreme points of $Y^{\leqq}$. The linear program (5.2) was first proposed and studied in Chapter 4. The next three results for problem (5.2) follow immediately from Chapter 4. For each result, the appropriate reference is given.

Theorem 5.3.1.1. (Theorem 4.2.2). Assume that $\bar{y} \in Y^{\leqq}$. If the optimal value of linear program (5.2) is less than 1.0, then $g(y^*) < g(\bar{y})$ for any optimal solution y^* of problem (5.2).

Theorem 5.3.1.2. (Theorem 4.3.5). Assume that $\bar{y} \in Y^{\leqq}$. If $\bar{y}$ is the unique optimal solution to linear program (5.2), then $\bar{y}$ is a strict local optimal solution for problem $(P_{Y^{\leqq}})$.

Theorem 5.3.1.3. (Theorem 4.3.6). Assume that $\bar{y} \in Y^{\leqq}$. If $\bar{y}$ is an optimal solution for linear program (5.2) and this linear program has multiple optimal solutions, then for any optimal solution $y^* \neq \bar{y}$ for problem (5.2), $g(y^*) < g(\bar{y})$ must hold.

From these results, we obtain the following procedure for finding a strict local optimal extreme point solution $\bar{y} \in Y^{\leqq}$ for problem $(P_{Y^{\leqq}})$ that satisfies $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in $Y^{\leqq}$.

Algorithm 5.3.1.1. Strict Local Optimal Solution Search Procedure.

Step 1. Find an initial extreme point $\bar{y}$ of $Y^{\leqq}$.

Step 2. Compute the optimal value v for linear program (5.2). If $v < 1.0$, go to Step 3. If $v = 1.0$, go to Step 4.

Step 3. Determine any optimal extreme point solution y^* for problem (5.2). Set $\bar{y} = y^*$ and return to Step 2.

Step 4. If problem (5.2) has multiple optimal solutions, go to Step 5. Otherwise, determine whether or not $g(y) < g(\bar{y})$ for some extreme point neighbor y to $\bar{y}$ in $Y^{\leqq}$. If so, find any such neighbor y, set $\bar{y} = y$, and return to Step 2. If not, then stop: The point $\bar{y}$ is a strict local optimal extreme point solution for problem $(P_{Y^{\leqq}})$ that satisfies $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in $Y^{\leqq}$.

Step 5. Find any optimal extreme point solution y^* for problem (5.2) that is distinct from $\bar{y}$. Set $\bar{y} = y^*$ and return to Step 2.

By using Proposition 5.2.1 and Theorems 5.3.1.1-5.3.1.3, it is easy to see that this search procedure is guaranteed to be finite and to find a strict local optimal extreme point solution $\bar{y}$ for problem $(P_{Y^{\leqq}})$ such that $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in $Y^{\leqq}$.

5.3.2. Cutting Plane Construction

Let $\bar{y}$ be a nondegenerate extreme point of $Y^{\leq}$ that is a strict local optimal solution for problem $(P_{Y^{\leq}})$ and satisfies $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in $Y^{\leq}$, and let $\gamma \in \boldsymbol{R}$ satisfy $\gamma \leqq g(\bar{y})$. Given $\bar{y}$, the new algorithm will seek to find a vector $\pi \in \boldsymbol{R}^p$ such that

$$\left\{y \in Y^{\leq} \middle| g(y) < \gamma\right\} \subseteq \left\{y \in Y^{\leq} \middle| \langle \pi, y - \bar{y} \rangle \geqq 1\right\}. \tag{5.3}$$

When π satisfies (5.3), the linear inequality $\langle \pi, y - \bar{y} \rangle \geqq 1$ is called a <u>γ-valid cutting plane</u> (or <u>γ-valid cut</u>) for problem $(P_{Y^{\leq}})$ (Horst and Tuy 1993).

Since $\bar{y}$ is a nondegenerate, strict local optimal extreme point solution for problem $(P_{Y^{\leq}})$ that satisfies $g(y) \geqq g(\bar{y})$ for each neighboring extreme point y to $\bar{y}$ in $Y^{\leq}$, since $Y^{\leq}$ has full dimension, and since g is strictly pseudoconcave on $\boldsymbol{R}_+^p$ (cf. Theorem 5.2.3), it follows that an approach used by Horst and Tuy (1993, pp. 85-91) can be used to find a vector $\pi \in \boldsymbol{R}^p$ that yields a γ-valid cut. The following procedure applies this approach to problem $(P_{Y^{\leq}})$.

<u>Algorithm 5.3.2.1.</u> Procedure for Cutting Plane Construction.

<u>Step 1.</u> Determine the p neighboring extreme points y^i, $i = 1, 2, \ldots, p$, to $\bar{y}$ in $Y^{\leq}$. Let $u^i = y^i - \bar{y}$, $i = 1, 2, \ldots, p$.

<u>Step 2.</u> For each $i \in \{1, 2, \ldots, p\}$ such that

$$\sup\left\{\theta_i > 0 \middle| g(\bar{y} + \theta_i u^i) \geqq \gamma\right\} \neq +\infty, \tag{5.4}$$

set $z^i = \bar{y} + \alpha_i u^i$, where α_i equals the value of the supremum in (5.4). For each $i \in \{1, 2, \ldots, p\}$ such that (5.4) does not hold, let $z^i = \bar{y} + \alpha_i u^i$, where $\alpha_i > 0$ is a finite number chosen so that $g(z^i) \geqq \gamma$.

Step 3. Compute $\pi^T = e^T Q^{-1}$, where $e \in R^p$ is the column vector of p ones, and Q is the $p \times p$ matrix with column i equal to $(z^i - \bar{y})$, $i = 1, 2, \ldots, p$, and stop: The linear inequality

$$\langle \pi, y - \bar{y} \rangle \geqq 1$$

is a γ-valid cutting plane for problem $(P_{\gamma\leqq})$.

Notice that in Step 1 of this procedure, since $\bar{y}$ is nondegenerate, the p neighboring extreme points described there are guaranteed to exist. Furthermore, because $Y^{\leqq}$ has full dimension, u^i, $i = 1, 2, \ldots, p$, are linearly independent (Murty (1983, pp. 138). Notice also in Step 2 that since $\bar{y}$ is a strict local optimal solution for problem $(P_{\gamma\leqq})$ for which $g(y^i) \geqq g(\bar{y})$, $i = 1, 2, \ldots, p$, for each $i = 1, 2, \ldots, p$, either $0 < \alpha_i < +\infty$ or a finite number θ_i can be chosen such that $g(\bar{y} + \theta_i u^i) \geqq g(\bar{y}) = \gamma$. The procedure is therefore well defined. By showing a result similar to Theorem III.1 in Horst and Tuy (1993), the procedure can also be shown to be valid.

When $\bar{y}$ is degenerate, since $Y^{\leqq}$ is full dimensional, $\bar{y}$ may have more than p neighboring extreme points y^i, $i = 1, 2, \ldots, s$, in $Y^{\leqq}$. In this case, Step 1 of the cutting plane construction procedure can be executed as stated, except with $p = s$, and, instead of Steps 2 and 3, the following steps can be executed (cf. Carvajal-Moreno 1972).

Step 2′ For each $i \in \{1, 2, \ldots, s\}$ such that (5.4) holds, set α_i equal the value of the supremum in (5.4). For each $i \in \{1, 2, \ldots, s\}$ such that (5.4) does not hold, let $\alpha_i > 0$ be a finite number such that $g(\bar{y} + \alpha_i u^i) \geqq \gamma$.

Step 3′. Let π be any basic solution to the system

$$\langle \pi, u^i \rangle \geqq (1/\alpha_i), \; i = 1, 2, \ldots, s, \tag{5.5}$$

and stop: The linear inequality

$$\langle \pi, y - \bar{y} \rangle \geqq 1$$

is a γ-valid cutting plane for problem $(P_{Y^{\leqq}})$.

Since $Y^{\leqq}$ has full dimension and $\bar{y}$ is an extreme point of $Y^{\leqq}$, by logic similar to that used in the proof of Lemma III.1 in Horst and Tuy (1993), it can be shown that in Step 3′, (5.5) has at least one basic solution. Furthermore, by showing a result similar to Proposition III.1 in Horst and Tuy (1993), the modifications to the cutting plane construction procedure described above can be shown to be appropriate for generating a γ-valid cutting plane for problem $(P_{Y^{\leqq}})$ when $\bar{y}$ is degenerate.

5.3.3. Termination Test

The linear programming-based termination test for the outcome-space, cutting-plane algorithm is contained in the following result.

Proposition 5.3.3.1. Let $\bar{y}$ be an extreme point of $Y^{\leqq}$ that is a strict local optimal solution for problem $(P_{Y^{\leqq}})$ and satisfies $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in $Y^{\leqq}$. Let

$$\langle \pi, y - \bar{y} \rangle \geqq 1$$

be a γ-valid cutting plane for problem $(P_{\gamma^{\leq}})$, where $g(\bar{y}) \geqq \gamma$ and, for some extreme point y^0 of $Y^{\leq}$, $\gamma = g(y^0)$. Then $\langle \pi, y - \bar{y} \rangle > 1$ for all $y \in Y^{\leq}$ such that $g(y) < \gamma$. Hence, if

$$1 \geqq \max \langle \pi, y - \bar{y} \rangle, \quad \text{s.t. } y \in Y^{\leq}, \tag{5.6}$$

then y^0 is a global optimal solution for problem $(P_{\gamma^{\leq}})$.

Proof. Since $\langle \pi, y - \bar{y} \rangle \geqq 1$ is a γ-valid cutting plane for problem $(P_{\gamma^{\leq}})$, we know from (5.3) that $\langle \pi, y - \bar{y} \rangle \geqq 1$ holds for all $y \in Y^{\leq}$ such that $g(y) < \gamma$. In fact, however, by logic similar to that used in the proofs of Theorem III.1 and Proposition III.1 in Horst and Tuy (1993), the strict inequality $\langle \pi, y - \bar{y} \rangle > 1$ can be shown to hold for all $y \in Y^{\leq}$ such that $g(y) < \gamma$. It is easy to see that this implies that if (5.6) holds, then y^0 is a global optimal solution for problem $(P_{\gamma^{\leq}})$. □

5.3.4. Outcome-Space, Cutting-Plane Algorithm

By incorporating the strict local optimal solution search procedure of Section 5.3.1, the cutting plane construction procedure of Section 5.3.2, and the termination test of Section 5.3.3 into the framework of a decision set-based, cutting plane method for concave minimization proposed by Horst and Tuy (1993, pp. 175-184), we obtain the following pure cutting plane algorithm for globally solving problem $(P_{\gamma^{\leq}})$. As shown in Section 5.2, from the global solution for problem $(P_{\gamma^{\leq}})$ obtained via this algorithm, a global optimal extreme point solution for problem (P_X) can be easily recovered by solving a single linear program.

Algorithm 5.3.4.1. Outcome-Space, Cutting-Plane Algorithm.

Initialization Step. By using the procedure in Section 5.3.1, find a strict local optimal extreme point solution $\bar{y}^0$ for problem $(P_{Y^{\leqq}})$ that satisfies $g(y) \geqq g(\bar{y}^0)$ for each extreme point neighbor y to $\bar{y}^0$ in $Y^{\leqq}$. Set $\gamma = g(\bar{y}^0)$ and $Y^0 = Y^{\leqq}$. Go to Step 0.

Step k, $k \geq 0$.

Step k.1. By using the procedure in Section 5.3.2, with $Y^{\leqq} = Y^k$, construct a γ-valid cut

$$\langle \pi^k, y - \bar{y}^k \rangle \geqq 1$$

for problem $(P_{Y^{\leqq}})$ at $\bar{y}^k$.

Step k.2. Set $\pi = \pi^k$, $\bar{y} = \bar{y}^k$, and $Y^{\leqq} = Y^k$ in the linear program in (5.6), and compute a basic optimal solution y^k to this linear program. If $\langle \pi^k, y^k - \bar{y}^k \rangle \leqq 1$, stop: The point $\bar{y}^0$ is a global optimal extreme point solution for problem $(P_{Y^{\leqq}})$, and γ is the global optimal value for problem $(P_{Y^{\leqq}})$. Otherwise, continue.

Step k.3. Let

$$Y^{k+1} = Y^k \bigcap \{ y \in R^p \mid \langle \pi^k, y - \bar{y}^k \rangle \geqq 1 \}.$$

Starting from y^k, use the procedure in Section 5.3.1 to find a strict local optimal extreme point solution $\bar{y}^{k+1}$ for the problem

$$\min g(y), \quad \text{s.t. } y \in Y^{k+1}$$

that satisfies $g(y) \geqq g(\bar{y}^{k+1})$ for each extreme point neighbor y to $\bar{y}^{k+1}$ in Y^{k+1}. If $g(\bar{y}^{k+1}) < \gamma$, then set $\gamma = g(\bar{y}^{k+1})$, $\bar{y}^0 = \bar{y}^{k+1}$ and $Y^0 = Y^{k+1}$, and go to Step 0. Otherwise, go to Step $k+1$.

The following result and proof are guided in part by Theorem V.2 in Horst and Tuy (1993).

Theorem 5.3.4.1. The outcome-space, cutting-plane algorithm for problem $(P_{Y^{\leqq}})$ is finite when the sequence $\{\pi^k\}$ is bounded. If the algorithm stops, the point $\bar{y}^0$ found by the algorithm is a global optimal extreme point solution for problem $(P_{Y^{\leqq}})$.

Proof. The algorithm consists of a number of groups (or cycles) of steps. The beginning of a typical cycle of steps occurs when, for some $k \geqq 0$, in Step k.3 the incumbent solution vector $\bar{y}^0$ is set equal to $\bar{y}^{k+1}$, the incumbent value γ is set equal to $g(\bar{y}^{k+1})$, and Y^0 is set equal to Y^{k+1}. During the cycle of steps, one or more γ-valid cuts is added to Y^0, but the incumbent solution $\bar{y}^0$ and the incumbent value γ remain unchanged. The cycle terminates when a point $\bar{y}^{k+1}$ is found such that $g(\bar{y}^{k+1}) < \gamma$. In view of Proposition 5.3.3.1, this implies that $\bar{y}^{k+1}$ strictly satisfies all of the γ-valid cutting planes created thus far by the algorithm. As a result, $\bar{y}^{k+1}$ is an extreme point of $Y^{\leqq}$, and it is distinct from all extreme points of $Y^{\leqq}$ previously encountered. By Proposition 5.2.1, $Y^{\leqq}$ has a finite number of extreme points. Therefore, the number of cycles in the algorithm must be finite.

During a typical cycle, one or more γ-valid cuts of the form

$$\langle \pi^h, y - \bar{y}^h \rangle \geqq 1,$$

where $h \geqq 0$ is an integer, is added to the current polyhedron $Y^0 \subseteq Y^{\leqq}$. For a typical $k \geqq 1$, from Step h.3 of the algorithm, we know that

$$\left\langle \pi^h, \bar{y}^k - \bar{y}^h \right\rangle - 1 \geqq 0, \tag{5.7}$$

$h = 0, 1, 2, \ldots, k-1$. We also note that

$$-1 = \left\langle \pi^k, \bar{y}^k - \bar{y}^k \right\rangle - 1 < 0. \tag{5.8}$$

From (5.7), (5.8) and Corollary III.2 in Horst and Tuy (1993), if $\{\pi^k\}$ is bounded, then the length of the cycle must be finite. Since the number of cycles in the algorithm is finite, this implies that when $\{\pi^k\}$ is bounded, the algorithm must be finite.

Now suppose that the algorithm stops. Then for some $k \geqq 0$, with $\pi = \pi^k$, $\bar{y} = \bar{y}^k$ and $Y^{\leqq} = Y^k$, inequality (5.6) becomes satisfied, after which the algorithm immediately stops. Furthermore, at the time the algorithm stops, it has found at the beginning of the current cycle of steps an extreme point $\bar{y}^0$ of $Y^{\leqq}$ and of Y^k that is the incumbent solution and satisfies $\gamma = g(\bar{y}^0) \leqq g(\bar{y}^k)$. In addition, $\bar{y}^k$ is a strict local optimal extreme point solution for the problem

$$\min g(y), \quad \text{s.t. } y \in Y^k$$

that satisfies $g(y) \geqq g(\bar{y}^k)$ for each extreme point neighbor y to $\bar{y}^k$ in Y^k. Taken together, these observations imply that we may apply Proposition 5.3.3.1 with $\pi = \pi^k$, $\bar{y} = \bar{y}^k$, $y^0 = \bar{y}^0$, and $Y^{\leqq} = Y^k$ to conclude that $\bar{y}^0$ is a global minimizer of g over Y^k. If $y \in Y^{\leqq}$ and $y \notin Y^k$, then Step k.3 implies that for some $\bar{\gamma}$-valid cut $\left\langle \pi^j, y - \bar{y}^j \right\rangle \geqq 1$, where $\bar{\gamma} \geqq \gamma$, $\left\langle \pi^j, y - \bar{y}^j \right\rangle < 1$ must hold. By (5.3), since $\gamma \leqq \bar{\gamma}$, this implies that $g(y) \geqq \bar{\gamma} \geqq \gamma$. Since $\gamma = g(\bar{y}^0)$, it follows that $g(y) \geqq g(\bar{y}^0)$ for all

$$y \in Y^k \bigcup \left\{ y \in Y^{\leqq} \middle| y \notin Y^k \right\},$$

i.e., $\bar{y}^0$ is a global optimal extreme point solution for problem $(P_{Y^{\leqq}})$. □

5.4. Implementation

In practice, by moderately modifying the procedures and the termination test in Sections 5.3.1-5.3.3, and by substituting Y for $Y^{\leqq}$ and the phrase "problem (P_Y)" for "problem $(P_{Y^{\leqq}})$" throughout the statement of the new outcome-space, cutting-plane algorithm, we obtain an algorithm that globally solves problem (P_Y) instead of problem $(P_{Y^{\leqq}})$. Since $Y \subseteq Y^{\leqq}$, computational savings can generally be expected to result by solving problem (P_Y) instead of problem $(P_{Y^{\leqq}})$. The validity of this approach relies in part upon the following two results.

Proposition 5.4.1. Let y^* be a global optimal solution for problem $(P_{Y^{\leqq}})$. Then $y^* \in Y$, and any $x^* \in X$ such that $Cx^* \leqq y^*$ satisfies $Cx^* = y^*$ is a global optimal solution for problem (P_X).

Proof. Since $Cx > 0$ for all $x \in X$, it is easy to see by the definition of $Y^{\leqq}$ that the global optimality of y^* in problem $(P_{Y^{\leqq}})$ implies that if $Cx^* \leqq y^*$ for some $x^* \in X$, then $Cx^* = y^*$. As a result, $y^* \in Y$, and, by Theorem 5.2.2, x^* is a global optimal solution for problem (P_X). □

A strict local optimal solution for problem (P_Y) is defined as in Definition 5.2.2, except with Y replacing $Y^{\leqq}$ in the definition.

Proposition 5.4.2. Problem (P_Y) possesses a global optimal solution that is an extreme point of Y. Any global optimal solution for problem (P_Y) must also be a strict local optimal solution for problem (P_Y).

Proof. From Rockafellar (1970), Y is a polyhedron. Since X is nonempty and compact, Y is also nonempty and compact. Furthermore, since $Cx > 0$ for all $x \in X$, $Y \subseteq \left\{ z \in R^p \middle| z > 0 \right\}$. Therefore, from Proposition 3.2.2, problem (P_Y) has a global optimal solution that is an extreme point of Y. A proof that any global optimal solution for problem (P_Y) must be a strict local optimal solution for problem (P_Y) is given by the proof of Theorem 5.2.4, except with Y replacing $Y^{\leqq}$ in the proof. □

Remark 5.4.1. From Proposition 5.4.1, we may confine the search for a global optimal solution for problem $(P_{Y^{\leqq}})$ to Y. Stated in a different way, any global optimal solution to problem (P_Y) is also a global optimal solution to problem $(P_{Y^{\leqq}})$. By Proposition 5.4.2, problem (P_Y) has a global optimal solution that is an extreme point of Y, and any global optimal extreme point solution to problem (P_Y) must be a strict local optimal solution for problem (P_Y). Taken together, since Y is a nonempty, compact polyhedron, these observations imply that to globally solve problem $(P_{Y^{\leqq}})$, we may apply the outcome-space, cutting-plane approach given in Section 5.3 to problem (P_Y), rather than to problem $(P_{Y^{\leqq}})$. Using this approach, we will find a global optimal solution to problem $(P_{Y^{\leqq}})$. By Proposition 5.4.1, a global optimal extreme point solution to problem (P_X) can then be recovered by solving a single linear program as explained in Section 5.2.

In addition to substituting Y for $Y^{\leqq}$ and the phrase "problem (P_Y)" for "problem $(P_{Y^{\leqq}})$" throughout the statement of the new outcome-space, cutting-plane algorithm in Section 5.3, some changes are needed to the procedures given in Sections 5.3.1 and 5.3.2 and in the termination test in Section 5.3.3 in order to confine the search to $Y \subseteq Y^{\leqq}$. Let us now explain these changes.

The following two results help to explain the changes needed in the strict local optimal solution search procedure given in Section 5.3.1 to confine the search to Y. The proof of the first result is immediate from the definitions.

Proposition 5.4.3. Any strict local optimal solution for problem $(P_{Y^{\leqq}})$ is also a strict local optimal solution for problem (P_Y).

Proposition 5.4.4. Assume that $\bar{y} \in Y$. Then any optimal solution for linear program (5.2) belongs to Y.

Proof. Suppose that y^* is an optimal solution to problem (5.2), but $y^* \notin Y$. Then, since $y^* \in Y^{\leqq}$ but $y^* \notin Y$, there exists some point $x^* \in X$ such that $Cx^* \leqq y^*$ but $Cx^* \neq y^*$. Let $y^{**} = Cx^*$. Then $y^{**} \in Y^{\leqq}$. Furthermore, since $\bar{y} > 0$, $y^{**} \leqq y^*$, and $y^{**} \neq y^*$, it follows that

$$(1/p)\sum_{j=1}^{p}\left(y_j^{**}/\bar{y}_j\right) < (1/p)\sum_{j=1}^{p}\left(y_j^{*}/\bar{y}_j\right).$$

Since $y^{**} \in Y^{\leqq}$, this inequality contradicts that y^* is an optimal solution to problem (5.2). Therefore, $y^* \notin Y$ cannot hold, and the proof is complete. □

From Propositions 5.4.3 and 5.4.4, if we replace $Y^{\leq}$ with Y in problem (5.2) and throughout the strict local optimal solution search procedure of Section 5.3.1, and if we replace the phrase "problem $(P_{Y^{\leq}})$" by "problem (P_Y)" in the procedure, then the modified procedure is guaranteed to find a strict local optimal extreme point solution $\bar{y}$ for problem (P_Y) that satisfies $g(y) \geqq g(\bar{y})$ for each extreme point neighbor y to $\bar{y}$ in Y. Thus, to apply the local search procedure in Section 5.3.1 to problem (P_Y), we need only to replace $Y^{\leq}$ with Y in linear program (5.2) and throughout the statement of the procedure, and to change the phrase "problem $(P_{Y^{\leq}})$" to "problem (P_Y)" in Step 4 of the procedure.

The cutting plane construction procedure given in Section 5.3.2 relies on the fact that at least p edges of $Y^{\leq}$ will emanate from every extreme point $\bar{y}$ of $Y^{\leq}$, where p is the dimension of $Y^{\leq}$. The dimension d of Y, however, may be less that or equal to p.

As a prerequisite to applying the cutting plane construction procedure when problem (P_Y) is being solved instead of problem $(P_{Y^{\leq}})$, a knowledge of the value of the dimension d of Y is required. A convenient way to find d is to insert the following step between Steps 1 and 2 of the strict local optimal solution search procedure given in Section 5.3.1.

Step 1′. Find all neighboring extreme points y^i, $i = 1, 2, \ldots, t$, to $\bar{y}$ in Y. Compute and save the rank d of the $t \times p$ matrix M, where row i of M is equal to $y^i - \bar{y}$, $i = 1, 2, \ldots, t$.

From Murty (1983, pp. 138), the value of d computed in Step 1′ equals the dimension of Y.

When $d = p$, the cutting plane construction procedure given in Section 5.3.2 can be directly used when solving problem (P_Y) by the new outcome-space, cutting-plane algorithm by simply replacing $Y^{\leqq}$ by Y and replacing the phrase "problem $(P_{Y^{\leqq}})$" with "problem (P_Y)" throughout the procedure. When $d < p$, a recent generalized cutting plane procedure given in Benson (1999) can be used instead of the procedure given in Section 5.3.2.

The termination test that is needed to adapt the outcome-space, cutting-plane algorithm for problem $(P_{Y^{\leqq}})$ to problem (P_Y) is obtained by replacing $Y^{\leqq}$ in (5.6) by Y. The validity of this change can be supported by proving a result for problem (P_Y) that is analogous to Proposition 5.3.3.1.

To implement the changes called for in this section to the new outcome-space, cutting-plane algorithm given in Section 5.3.4, mechanics must be available for finding an initial extreme point of Y and, given an extreme point $\bar{y}$ of Y, for finding all neighboring extreme points in Y to $\bar{y}$. Given these mechanics, one can then carry out all of operations called for by the revised outcome-space, cutting-plane algorithm, including the solution of all of the required linear programming problems over Y and the construction and incorporation into Y of all required γ-valid cuts.

A finite linear programming-based algorithm guaranteed to find an initial extreme point of Y is given in Benson (1998d). In Benson and Sun (1998), the mechanics are given for finding all extreme points of Y that are adjacent to a given extreme point of Y.

From the previous paragraph, the existence of these procedures guarantees that the outcome-space, cutting-plane algorithm for globally solving problem (P_Y) can, indeed, be implemented in practice. As a result, as we have seen, a global optimal extreme point solution for problem (P_X) can be found.

5.5. Example

To illustrate the application of the outcome-space, cutting-plane algorithm to problem (P_X), consider the case with $p=2$ and $X=\left\{x\in \boldsymbol{R}^{11}\middle| Ax=b, x\geqq 0\right\}$, where

$$A=\begin{bmatrix} 9 & 9 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 1 & 8 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 8 & 8 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 7 & 1 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 7 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 1 & 1 & 7 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad b=\begin{bmatrix} 81 \\ 72 \\ 72 \\ 9 \\ 9 \\ 9 \\ 8 \\ 8 \end{bmatrix},$$

and where

$$c_1=\begin{bmatrix} 1 \\ 0 \\ 1/9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad c_2=\begin{bmatrix} 0 \\ 1 \\ 1/9 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

It can be shown that, in this case, X is a nonempty, compact polyhedron, and $Cx > 0$ for all $x \in X$, as required in problem (P_X). To solve problem (P_X), as explained previously, we will apply the modified outcome-space, cutting-plane algorithm to problem (P_Y). In this case, problem (P_Y) is given by

$$\begin{aligned} &\min\ y_1 y_2, \\ &\text{s.t.} \\ &\quad \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} x_1 + (1/9)x_3 \\ x_2 + (1/9)x_3 \end{bmatrix} \text{ for some } x \in X. \end{aligned}$$

A summary of the application of the modified outcome-space, cutting-plane algorithm to problem (P_Y) in this case is given below. The mechanics used on the set Y, including the generation of an initial extreme point of Y and the solution of linear programming problems with feasible regions given by Y, are taken from Benson (1998d) and Benson and Sun (1998).

Initialization Step. We execute the search procedure given in Section 5.3.1, but adapted to problem (P_Y). The initial extreme point $\bar{y}$ of Y that we find thereby is given by $\bar{y}^T = (1/9, 8\ 1/9)$. As called for by Step 1′ in Section 5.4, we find that the extreme point neighbors to $\bar{y}$ in Y are $(y^1)^T = (1,1)$ and $(y^2)^T = (9/10, 8\ 1/10)$. The matrix M is thus given by

$$M = \begin{bmatrix} 8/9 & -7\ 1/9 \\ 71/90 & -1/90 \end{bmatrix}.$$

We set $d = \text{rank}\, M = 2$, and we save $d = 2$ for future reference. Next, with Y replacing $Y^{\leqq}$ in (5.2), the optimal value of linear program (5.2) is found to be $v = 1.0$, and we find that $\bar{y}$ is the unique optimal solution to this problem. Therefore, we compare

$g(\bar{y}) = \bar{y}_1\bar{y}_2 = 73/81$ to the values of g evaluated at the neighboring extreme points y^1 and y^2 to $\bar{y}$ in Y. We find that

$$g(\bar{y}) = 73/81 < g(y^1) = 1.0$$

and

$$g(\bar{y}) = 73/81 < g(y^2) = 7.29.$$

The strict local optimal extreme point search is therefore terminated, and we set $\bar{y}^0 = \bar{y}$, where $\bar{y}^T = (1/9, 8\ 1/9)$. We set $\gamma = g(\bar{y}^0) = 73/81$, $Y^0 = Y$, and proceed to Step 0.

Step 0.

Step 0.1. We construct a γ-valid cut for problem (P_Y) with $Y = Y^0$ at $\bar{y}^0$. We accomplish this by executing the procedure given in Section 5.3.2, but adapted to problem (P_Y) with $Y = Y^0$. In particular, we first determine whether $\bar{y}^0$ has exactly d or more than d neighboring extreme points in Y. In this case, we know from the initialization step that $\bar{y}^0$ has exactly $d = 2$ neighboring extreme points in Y. Therefore, $\bar{y}^0$ is nondegenerate, and we will execute the cutting plane procedure given in Section 5.3.2 for the nondegenerate case, modified for problem (P_Y). Towards this end, we set $(u^1)^T = (8/9, -7\ 1/9)$ and $(u^2)^T = (71/90, -1/90)$. Next, we compute

$$\sup\{\theta_i > 0 \mid g(\bar{y} + \theta_i u^i) \geq \gamma\}, \quad i = 1, 2,$$

and we find that for $i = 1$, this supremum is finite and equals 1.015625 and, for $i = 2$, this supremum is finite and equals 729.85915. Therefore, we set $\alpha_1 = 1.015625$ and $\alpha_2 = 729.85915$. Then, for each $i = 1, 2$, we set $z^i = \bar{y} + \alpha_i u^i$, which results in the values

$$\left(z^1\right)^T = (1.013888, 0.888888),$$

$$\left(z^2\right)^T = (575.888888, 0.001565).$$

Next, we compute $\left(\pi^0\right)^T = e^T Q^{-1}$, where

$$Q = \begin{bmatrix} 0.902777 & 575.777777 \\ -7.222222 & -8.109546 \end{bmatrix}.$$

This yields $\left(\pi^0\right)^T = (-0.0002138, -0.1384883)$. The γ-valid cut, after gathering terms, is then given by

$$0.0002138\, y_1 + 0.1384883\, y_2 \leqq 0.1233175.$$

Step 0.2. With $\pi = \left(\pi^0\right)^T$, $\bar{y} = \bar{y}^0$, and $Y^{\leqq} = Y^0$, the linear program in (5.6) is found to have y^0 as a basic optimal solution, where

$$\left(y^0\right)^T = (73/9, 1/9),$$

and an optimal value of 1.1064. Since $1.1064 > 1.0000$, we continue to Step 0.3.

Step 0.3. We first set

$$Y^1 = Y^0 \bigcap \left\{ y \in R^2 \middle| 0.0002138\, y_1 + 0.1384883\, y_2 \leqq 0.1233175 \right\}.$$

Next, starting from y^0, we execute the search procedure given in Section 5.3.1, but adapted to problem (P_γ) with $Y = Y^1$, in order to find a strict local optimal extreme point solution $\bar{y}^1$ for the problem

$$\min\ g(y), \quad \text{s.t. } y \in Y^1$$

that satisfies $g(y) \geqq g\left(\bar{y}^1\right)$ for each extreme point neighbor y to $\bar{y}^1$ in Y^1. This results in

$$\left(\bar{y}^1\right)^T = (8\ 1/9, 1/9).$$

Since $g(\bar{y}^1) = 73/81 < \gamma$, we proceed to Step 1.

Step 1.

Step 1.1. We construct a γ-valid cut for problem (P_Y) with $Y = Y^1$ at $\bar{y}^1$. To do so, we use the procedure in Section 5.3.2, but adapted to problem (P_Y) with $Y = Y^1$. We find that $\bar{y}^1$ has exactly $d = 2$ neighboring extreme points in Y^1, so that $\bar{y}^1$ is nondegenerate. The resulting γ-valid cut is, after gathering terms,

$$0.1384833 y_1 + 0.0002138 y_2 \leqq 0.1233175.$$

Step 1.2. With $\pi = \pi^1$, $\bar{y} = \bar{y}^1$, and $Y^{\leqq} = Y^1$, we find that the linear program in (5.6) has the point y^2 given by

$$(y^2)^T = (1.9003173, 0.8874603)$$

as a basic optimal solution, and that it has an optimal value of 0.8599391. Since $0.8599391 < 1.0$, the algorithm stops, since it has found the global optimal solution $\bar{y}^0$ for problem (P_Y), where

$$(\bar{y}^0)^T = (1/9, 8\ 1/9),$$

and $\gamma = 73/81$ is the global optimal value for problem (P_Y).

As explained in Remark 5.4.1 and Section 5.2, by solving a single linear program, an extreme point global optimal solution to problem (P_X) can be recovered from $\bar{y}^0$ in the sample problem. By solving this linear program, we obtain the extreme point global optimal solution $(x^*)^T = (0.0, 8.0, 1.0)$ for problem (P_X).

5.6. Concluding Remarks

Both theoretical and empirical evidence have shown that generally an outcome polyhedron can be expected to have a significantly lower dimension, a significantly simpler structure, and far fewer extreme points than the underlying decision set polyhedron. For the case of the linear multiplicative program (P_X), we have in this chapter constructed and validated an outcome-space, cutting-plane algorithm that searches the outcome polyhedron, rather than the decision set polyhedron, for a global optimal extreme point solution. We expect, therefore, that the algorithm proposed here for problem (P_X) will potentially have computational benefits over the decision set-based algorithms that have been proposed for the problem.

CHAPTER 6
SUMMARY AND FUTURE RESEARCH

6.1. Introduction

In this dissertation, we have analyzed the multiplicative programming problem (P_X). For the linear case of problem (P_X), we used these analyses to develop a heuristic algorithm that finds a good solution to the problem and to present a global solution algorithm for the problem. In this chapter, we will examine potential avenues for future research that might potentially improve the efficiency and accuracy of the heuristic algorithm. We also examine possible avenues for potential global optimal solution algorithms for problem (P_X) based on analytical results we have presented.

6.2. Future Research on the Heuristic Algorithm

In preparation for presenting our heuristic algorithm for the linear case of problem (P_X) in Chapter 3, we analyzed the case when problem (P_X) is the concave multiplicative programming problem. We showed that when problem (P_X) is a concave multiplicative programming problem with a compact feasible region, it has at least one global optimal solution that is an efficient extreme point of the associated multiple-objective programming problem

$$(MOMP) \qquad \text{VMIN } f(x), \text{ s.t. } x \in X,$$

where $f_j : X \to R,\ j = 1, 2, \ldots, p,$ are the functions used in defining problem (P_X) and where $f(x)$ denotes the vector

$$[f_1(x), f_2(x), \ldots, f_p(x)]^T.$$

Our heuristic algorithm for the linear multiplicative program with a compact polyhedral feasible region, uses this efficiency result to search for a good solution by limiting the search to the efficient set of the multiple-objective linear program

$$(MOLP) \qquad \text{VMIN } Cx, \text{ s.t. } x \in X,$$

where C is the $p \times n$ matrix whose jth row equals $(c^j)^T$, $j = 1, 2, \ldots, p$. The heuristic algorithm first identifies an efficient face X_w of problem (*MOLP*). It then searches the efficient face X_w for an good solution to problem (P_X) by optimizing a linear approximation of the objective function of problem (P_X) over X_w. Our experimental results show that the computational requirements of the heuristic algorithm are not overly burdensome when compared to the effort required to solve a linear multiplicative programming problem.

The performance of the heuristic algorithm depends in part upon the number, locations, and dimensions of the efficient faces found in the Efficient Point Search Phase. The user can manipulate the location and number of efficient faces found by selecting values for the parameters *N*, used for the weighting of the functions in the objective function, and *S*, the number of sample objective function values from $Y^{\geq}$. A topic for further research would be to identify criteria and methods for choosing good values for these parameters so that a variety of regions of the efficient set are searched for a good solution for problem (P_X).

In Step 1 of the Efficient Point Search Phase, an efficient extreme point is used to identify an efficient face X_w on which that extreme point lies. Currently, there is no

known procedure that can be used to find an efficient face of maximal dimension on which an extreme point lies. Since the performance of the heuristic algorithm can be improved by searching maximal efficient faces for good solutions, a topic for further research would be to find such a procedure.

6.3. Future Research on an Global Solution Algorithms

In Chapter 3 we showed that a concave multiplicative programming problem can be solved as a quasiconcave minimization problem or reformulated as a concave minimization problem. We also showed that a concave multiplicative programming problem with a compact feasible region has a globally optimal solution that is an efficient extreme point of the associated multiple-objective programming problem (*MOMP*). A potential opportunity therefore exists to improve the efficiency of a concave minimization method for solving the concave multiplicative programming problem or the linear multiplicative programming problem by limiting the search to the efficient extreme points of the associated multiple-objective programming problem (*MOMP*) or (*MOLP*) respectively. For example, for the linear case of problem (P_X), enumerative concave minimization methods such as extreme point ranking or cutting plane methods can be modified to limit the search to those extreme points of problem (*MOLP*) that are efficient. For the concave case of problem (P_X), outer approximation and branch and bound methods may be modified to limit the search to efficient extreme points of problem (*MOLP*). For more information on concave minimization methods, see, for example, Benson (1995) and Horst and Tuy (1993).

In preparation for developing our outcome-space, cutting-plane algorithm for the linear multiplicative programming problem in Chapter 4, we analyzed the outcome-space reformulation problem $(P_{Y^{\leqq}})$ of the multiplicative programming problem (P_X). Since the reformulation $(P_{Y^{\leqq}})$ works in a lower dimensioned space than the space of the feasible decision set X, potential computational savings can be expected by solving problem $(P_{Y^{\leqq}})$ for an optimal solution $y^* \in Y^{\leqq}$. As we have shown, a global optimal solution $x^* \in X$ for problem (P_X) can then be recovered from y^*.

In Chapter 4 we showed that when problem (P_X) is a convex multiplicative programming problem, a global optimal solution y^* of the reformulation $(P_{Y^{\leqq}})$ will exist on the boundary of the convex outcome set $Y^{\leqq}$. In addition, when the convex set $Y^{\leqq}$ is closed and has at least one extreme point, our analysis showed that problem $(P_{Y^{\leqq}})$ has at least one global optimal solution that is an extreme point of $Y^{\leqq}$. We also showed that the objective function of problem $(P_{Y^{\leqq}})$ is strictly pseudoconcave over $Y^{\leqq}$. Since pseudoconcave functions defined over convex sets are also quasiconcave functions, many of the most popular algorithms for minimizing concave functions over a convex sets are equally suitable for minimizing quasiconcave functions over convex sets (Horst and Tuy 1993 and Benson 1995). Our results thus provide possible avenues for proposing implementable solution methods for the convex multiplicative programming problem that solve instead the reformulation $(P_{Y^{\leqq}})$ using existing concave minimization methods. For example, outer approximation can be used to search the boundary of $Y^{\leqq}$ for a global optimal y^* solution for problem $(P_{Y^{\leqq}})$. This method cannot be applied directly to the

original convex multiplicative programming problem since, as shown in Example 4.3.1, the preimage $x^* \in X$ of y^* need not lie on the boundary of X. For more information on concave minimization methods see, for example, Benson (1995) and Horst and Tuy (1993).

In Chapter 5 we presented the outcome-space, cutting-plane algorithm for solving the case when problem (P_X) is the linear multiplicative programming problem. It solves the reformulation problem $(P_{Y^{\leq}})$ of problem (P_X) for a global optimal extreme point solution $y^* \in Y^{\leq}$ for problem $(P_{Y^{\leq}})$, and then recovers a global optimal extreme point solution $x^* \in X$ of problem (P_X). The reformulation $(P_{Y^{\leq}})$ of the linear multiplicative programming problem (P_X) offers another computational advantage in addition to that of working in a lower dimensioned space than the space of the feasible region X. A large number of extreme points of X collapse into nonfacial structures or into the relative interiors of subfaces of the outcome set $Y^{\leq}$, but not vice versa (Benson 1995a). Since the outcome-space, cutting-plane algorithm searches only those extreme points of $Y^{\leq}$ that are the image of extreme points and faces of X under the mapping of the functions in the objective function of problem (P_X), many non-optimal extreme points of X are implicitly bypassed in the algorithm's search. We therefore expect that the outcome-space, cutting-plane algorithm will potentially have computational benefits over the decision set-based algorithms that have been proposed for the linear multiplicative programming problem.

To aid in the implementation of the outcome-space, cutting-plane algorithm, the algorithm can be implemented using only linear programming methods. A topic for

further research is to code the algorithm and test its performance against other algorithms for solving linear multiplicative programming problems.

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BIOGRAPHICAL SKETCH

George Boger received his Bachelor of Science in Mathematics in 1973 from the University of Central Florida. Following graduation he worked in the federal civil service for NASA and the U.S. Navy. In 1992 he received a Master of Science in Operations Research from the Florida Institute of Technology and entered the Ph.D. program in the Department of Decision and Information Sciences at the University of Florida. After graduation, George plans to work in academia, teaching and conducting research.

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Harold P. Benson, Chairman
Professor of Decision and
Information Sciences

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S. Selcuk Erenguc
Professor of Decision and
Information Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Asoo J. Vakharia
Associate Professor of Decision and
Information Sciences

I certify that I have read this study and that in my opinion it conforms to acceptable standards of scholarly presentation and is fully adequate, in scope and quality, as a dissertation for the degree of Doctor of Philosophy.

Richard L. Francis
Professor of Industrial and Systems
Engineering

This dissertation was submitted to the Graduate Faculty of the Department of Decision and Information Sciences in the College of Business Administration and to the Graduate School and was accepted as partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Dean, Graduate School

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